Median Confidence Intervals

Extended Abstract

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INTRODUCTION

Stochastic simulation profits from fast computers, many more and much longer simulation runs can be carried out in reasonable time than a decade ago, estimation can rely on many data. This incrases the trustworthiness in the statistical simulation results, and in many situations, the distributions of estimators can be expected to be close to a normal distribution, in which the expectation and the median coincide. This observation motivates us to consider confidence intervals around the median as a substitute of confidence intervals (CI). These median confidence intervals (MCI) are easier to obtain than usual confidence intervals: No variance of the estimator is needed, and it is straightforward to get MCIs for functions of estimators.

CONFIDENCE INTERVALS

Let $X_1, X_2, ..., X_n$ be IID random variables with finite mean μ and finite variance σ^2. If the X_is are normal random variables, or if n is sufficiently large the sample mean

$$\bar{X}(n) = (X_1 + ... + X_n)/n$$

is (approximately) a normal random variable. Given the confidence level $1 - \alpha$,

$$\left[\bar{X}(n) - z_{1-\alpha/2}\sqrt{\sigma^2/n},\ \bar{X}(n) + z_{1-\alpha/2}\sqrt{\sigma^2/n}\right]$$

is a confidence interval where $z_{1-\alpha/2}$ is the upper critical value of the standard normal distribution, $\Phi(z_{1-\alpha/2}) = 1 - \alpha/2$ holds for the standard normal distribution function $\Phi(z)$.

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The variance σ^2 can be replaced by the sample variance

$$S^2(n) = \frac{\sum_{i=1}^{n}(X_i - \bar{X}(n))^2}{n-1}$$

and here the confidence interval is

$$\left[\bar{X}(n) - t_{n-1,z_{1-\alpha/2}}\sqrt{S^2(n)/n},\ \bar{X}(n) + t_{n-1,z_{1-\alpha/2}}\sqrt{S^2(n)/n}\right]$$

where $t_{n-1,z_{1-\alpha/2}}$ is the upper critical value for the t distribution with $n-1$ degrees of freedom. If n is large, $t_{n-1,z_{1-\alpha/2}} \approx z_{1-\alpha/2}$ holds.

Now let $\bar{X}_1, ..., \bar{X}_w$ be IID random variables with finite mean μ and finite median $\tilde{\mu}$, hence $P\{\bar{X}_i < \tilde{\mu}\} \leq 0.5$ and $P\{\bar{X}_i > \tilde{\mu}\} \leq 0.5$. With $\bar{X}^{\min} = \min_{1\leq i\leq w} \bar{X}_i$ and $\bar{X}^{\max} = \max_{1\leq i\leq w} \bar{X}_i$, $P\{\bar{X}^{\min} < \tilde{\mu} < \bar{X}^{\max}\} \leq 1 - 0.5^{w-1}$ holds.

$$\left[\bar{X}^{\min},\ \bar{X}^{\max}\right]$$

is the median confidence interval for the confidence level $1-\alpha$ with $\alpha = 0.5^{w-1}$. We consider it as a substitute for a confidence interval if the median equals (approximately) the mean, $\mu = \tilde{\mu}$ or $\mu \approx \tilde{\mu}$. For some distributions which occur frequently in simulation, this can be assumed, for example the normal and the uniform distribution have $\mu = \tilde{\mu}$, and $\mu \approx \tilde{\mu}$ holds for the t and the χ^2 distribution if the degree of freedom is high.

In simulation, the above $\bar{X}_i$s are estimators in w independent runs (replications).

Example

Let $X_{1,1}, ..., X_{1,n}, X_{2,1}, ..., X_{2,n}, ..., X_{w,1}, ..., X_{w,n}$ be independent $N(\mu, \sigma^2)$-distributed random variables (normal distribution with mean μ and variance σ^2). The mean μ is to be estimated, the variance σ^2 is known.

With the estimator $\bar{X}(wn) = (X_{1,1} + ... + X_{w,n})/(wn)$,

$$\left[\bar{X}(wn) - z_{1-\alpha/2}\sqrt{\sigma^2/(wn)},\ \bar{X}(wn) + z_{1-\alpha/2}\sqrt{\sigma^2/(wn)}\right]$$

is a confidence interval with confidence level $1-\alpha$.

With the independent erstimators $\bar{X}_i(n) = X_{i,1} + ... + X_{i,n})/n, \ \ i = 1, ..., w$, the median confidence interval is $\left[\bar{X}^{\min},\ \bar{X}^{\max}\right]$; the confidence level is $1-\alpha$ where $\alpha = 1/2^{w-1}$. $\square$

Median confidence intervals have attractive features.

1. The variance of the estimator is not needed. Here is a main difficulty when confidence intervals are constucted because "simulation output data are always correlated" (Law and Kelton [3]). Special procedures must be applied for this variance, the replication/deletion appoach [3], batch means [2, 3], the regenerative method [2, 3], autoregressive processes [2, 3], the spectral estimation method [2, 3], or the standadized time series method [3], all of which are not free from obstacles. This is omitted for median confidence intervals.

2. It is easy to obtain median confidence intervals for functions of some different estimators whereas it is difficult to get confidence intervals, in general. For example, in a queueing system, the mean waiting time in the queue, $E[W]$, and the arrival rate λ are estimated. The number of jobs in the queue is $\lambda E[W]$ (Little's Law). One can give a median confidence interval as above but the confidence interval needs more care.

3. If in a simulation a median confidence interval is too wide the replications are too short. A smaller median confidence interval can be obtained when the w replications are augmented, the runs are continued with $X_{i,n+1}$.

4. The median confidence intervals are only slightly wider than confidence intervals, but not too much. In some investigation, we found mean ratios between 1.34 and 1.52, see next pages. The reader may observe that the width of median confidence intervals are random variables whereas it is constant for confidence intervals.

WIDTH OF CONFIDENCE INTERVALS

Now we compare the widths of confidence intervals and median confidence intervals by means of an example which is not too special. In a simulation with w replications, the estimators $\bar{X}_1, ..., \bar{X}_w$ are $N(\mu, \sigma^2/n)$-distributed random variables. For example they are sample means of many random variables which may be correlated. Given the confidence level $1-\alpha$ where $\alpha = 1/2^{w-1}$, and $\bar{X} = (\bar{X}_1 + ... + \bar{X}_w)/w$,

$$\left[\bar{X} - z_{1-\alpha/2}\sqrt{\sigma^2/(wn)},\ \bar{X} + z_{1-\alpha/2}\sqrt{\sigma^2/(wn)}\right]$$

is a confidence interval for μ, the width b^{CI} of which is $2z_{1-\alpha/2}\sqrt{1/w}\sqrt{\sigma^2/n}$. The median confidence interval with the same confidence level and the same number of random numbers is

$$[\min \bar{X}_i,\ \max \bar{X}_i].$$

$Y_i = (\bar{X}_i - \mu)\sqrt{n/\sigma^2}$ are $N(0,1)$-distributed random variables. The width of the median confidence intervals is given by the random variable

$$B^{\mathrm{MCI}} = (\max_{1\leq i\leq w} Y_i - \min_{1\leq i\leq w} Y_i)\sqrt{\sigma^2/n}.$$

We are interested in the ratio $r = E[B^{\mathrm{MCI}}]/b^{\mathrm{CI}}$,

$$r = \frac{E[\max_{1\leq i\leq w} Y_i - \min_{1\leq i\leq w} Y_i]}{2z_{1-\alpha/2}\sqrt{1/w}}$$

because it is a measure for the quality of median confidence intervals.

Above we mentioned that we found ratios of 1.34...1.52. This was accomplished as follows. The distribution function of $\max_{1\leq i\leq w} Y_i - \min_{1\leq i\leq w} Y_i$ can be expressed with products of normal distribution functions and a convolution, but it seems that it is not easy to calculate the expectation. This is why we estimated $E[\max_{1\leq i\leq w} Y_i - \min_{1\leq i\leq w} Y_i]$ with simulation as follows.

In each of six replications, n' figures $b_{k,j} = \max_{1\le i\le w} y_i - \min_{1\le i\le w} y_i,\ k = 1,...,6,\ j = 1,...,n'$, are calculated where $y_i,\ i = 1,...,w$, are $N(0,1)$-distributed random numbers. With the six replications the confidence level for MCIs is $1 - 1/2^5$. The same confidence level is adopted for confidence intervals – of course we calculated both, MCIs and CIs. The mean

$$\bar{b}(6n') = \frac{1}{6n'} \sum_{k=1}^{6} \sum_{j=1}^{n'} b_{k,j}$$

estimates the expectation $E[\max_{1\le i\le w} Y_i - \min_{1\le i\le w} Y_i]$. A concrete confidence interval is

$$\left[\bar{b}(6n') - z_{1-1/2^5} \sqrt{\frac{1}{6n'}},\ \bar{b}(6n') + z_{1-1/2^5} \sqrt{\frac{1}{6n'}} \right]$$

and a concrete median confidence interval

$$\left[\min_{1\le k\le 6} \bar{b}_k(n'),\ \max_{1\le k\le 6} \bar{b}_k(n') \right]$$

where

$$\bar{b}_k(n') = \frac{1}{n'} \sum_{j=1}^{n'} b_{k,j},\ k = 1,...,6.$$

The following table presents some simulation results: The estimated expectation $e = E[\max_{1\le i\le w} Y_i - \min_{1\le i\le w} Y_i]$ with confidence intervals and median confidence intervals where the confidence level is $1 - \alpha$ with $\alpha = 1/2^5 = 0.03125$. In the last column, the resulting ratio r is given. Obviously it depends on w, the number of replications.

w	n'	Expectation e	MCI for e	CI for e	Ratio r
4	1600	2.045	[2.0084, 2.0643]	[2.0257, 2.0644]	1.33
4	409200	2.058	[2.0571, 2.0595]	[2.0572, 2.0596]	1.34
6	1600	2.534	[2.5123, 2.5602]	[2.5151, 2.5524]	1.44
6	409200	2.535	[2.5335, 2.5367]	[2.5338, 2.5362]	1.44
11	1600	3.165	[3.1537, 3.1766]	[3.1476, 3.1822]	1.52
11	409600	3.173	[3.1720, 3.1749]	[3.1723, 3.1745]	1.52

Altogether, we did 23 simulations; all results are in accordance with those in the table.

These simulations were done for the estimation of the width of MCIs for different numbers w of replications. Now we interpret these 23 simulations as examples for the MCI technique.

All MCIs were calculated with 6 replications. Hence we expect the ratio length(MCI)/length(CI) to be something like 1.44. Indeed, we found ratios between 0.624 and 2.45, the mean was 1.358.

In the 23 simulations we draw millions of $(\bar{y}_1, ..., \bar{y}_w)$-tuples, each of which represents a simulation and a MCI. A tuple with $\min \bar{y}_i > 0$ or with $\max \bar{y}_i < 0$ corresponds to such a simulation where the estimated μ lies outside of the MCI. We counted how often this occured and found very good accordance with the confidence level, more precisely with α. The following table containes typical examples:

w	α	Number of Simulations	Counted Fraction
4	12.5%	600	12.666%
4	12.5%	153600	12.39%
6	3.125%	600	3.67%
6	3.125%	153600	3.16%
11	0.0976%	600	0.167%
11	0.0976%	153600	0.102%

In [4] we present a simulation technique for stiff Markov chains, with other words with rare events, which is based upon Courtois' method [1]. Here, a product of expectations is to be estimated, namely $P\{R\}P\{S|R\}$. $P\{R\}$ is the probability that the system state belongs to the set of rare states, R, and $P\{S|R\}$ the probability that the system is in the rare state S conditioned that it is in a rare state. For these products we determined MCIs with 6 replications. If an MCI was too wide, we continued the runs. This was done until the MCIs were small enough.

In the paper, we will look closer on the analysis of dependent simulation data. Median confidence intervals are compared with confidence intervals.

References

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[4] J. Ch. Strelen. Monte-Carlo simulation of Markov chains using a high-level modelling technique. In A. Bargiela and E. Kerkhoffs, editors, *Simulation Technology: Science and Art - Proceedings of the ESS 98*, pages 213–217. Society for Computer Simulation, 1998.