# A Little Tutorial on the DA Iteration Method: Three Tiny Blocking Networks Solved Step by Step

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**Abstract**

The steps of numerical calculation using our disaggregation-aggregation (DA) method are explained in a tutorial fashion. This should train the reader to applying the DA method. The text is not self-contained - the knowledge of the DA method is required - rather it is a supplement *.

## 1 Queueing networks with repetitive-service blocking

We consider queueing networks with repetitive service blocking [3]. A network consists of $K \in \mathbb{N}$ nodes. In node $k$, $k = 1, \ldots, K$, there is a single exponential server, service rate $\mu_k$. Buffer space is available for no more than $\nu_k$ jobs. If the model is open, jobs arrive according to a Poisson process with parameter $\lambda_k$. When a service of a job in node $k$ is finished, the job tries to go to node $l$ with transfer probability $t_{k,l}$, $l \in [1 : K]$. If there is no buffer space available, the job remains in node $k$ and instantaneously receives another service (*repetitive-service with random destination policy*). In an open network, a job which is completely served in node $k$ leaves the network with probability $t_{k,0}$. In a closed network, there are $N \in \mathbb{N}$ jobs, and we assume without loss of generality $\nu_k \leq N$, $k = 1, \ldots, K$.

The system's state is given by the $K-$tuple $\mathbf{z} = (z_1, \ldots, z_K)$ where $z_k$ denotes the number of jobs in node $k$. In an open queueing network, the state space

$$\mathcal{Z}^T = \mathcal{S}_1 \times \ \ldots \ \times \mathcal{S}_K$$

where $\mathcal{S}_k = [0 : \nu_k]$, and in a closed queueing network

$$\mathcal{Z}^T = \{\mathbf{z} \in \mathcal{S}_1 \times \ \ldots \ \times \mathcal{S}_K \mid z_1 + \ldots + z_K = N\},$$

*The prerequisite papers can be read at www.cs.uni-bonn.de/II/staff/strelen/SelectedPapers.htm
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where

$$\mathcal{S}_k = [\tau_k : \nu_k], \quad \tau_k = \max\Big(0, N - \sum_{j \neq k} \nu_j\Big) :$$

A state $\tau_k > 0$ in node $k$ occurs if no buffer space is left in any other node. □

The probabilities $B_k$ that a job is blocked in node $k$, the *blocking probabilities*, are given by

$$B_k = \sum_{\substack{j=1 \\ j \neq k}}^{K} t_{k,j} \Big( \mathrm{P}\{Z_j = \nu_j\} - \mathrm{P}\{Z_k = 0,\ Z_j = \nu_j\} \Big), \quad k = 1, \ldots, K$$

($Z_k$ is the state random variable of component $k$).

In closed networks, the transitions from state $\mathbf{y} \in \mathcal{Z}^T$ into state $\mathbf{z} \in \mathcal{Z}^T$ occur as follows: Service at node k is completed, the job goes from node $k$ to another node. In open networks, additionally arrivals of jobs may occur, and jobs may leave the network after a service.

The state transitions can be expressed in the generator matrix, [3, (17)].

Transition classes (TC) [2] describe state transitions more concise. Here they are as follows: The system may change its state due to an arrival at node $k$ from outside. The source state set of this transition class is $\mathcal{U}_\nu = \{\mathbf{z} \in \mathcal{Z}^T \mid z_k < \nu_k\}$. That means, there is space left in node $k$ for an arriving job. Only the state of node $k$ changes, the other remain unchanged. Hence the destination state function is given by $u_{\nu,k}(\mathbf{z}) = z_k + 1$, identity otherwise: $u_{\nu,i}(\mathbf{z}) = z_i$ if $i \neq k$. The transition probability is $\alpha_\nu = \lambda_k \Delta$.

$\Delta$ is the randomization constant: Initially, the Markov chain is continuous in time. Let $\mathbf{Q}$ denote its generator matrix. We consider the discrete time Markov chain with the transition probability matrix $\mathbf{P} = \mathbf{Q}\Delta + \mathbf{I}$ instead, where $\Delta = 0.99 / \max_x |q_{x,x}|$. It has the same steady state probabilities as the continuous time Markov chain. By the factor 0.99, numerical problems are avoided, which may occur due to rounding errors.

Other state transitions take place when a job of node $k$ leaves the network. The according transition class $\nu$ is defined as follows. The source states set is

$$\mathcal{U}_\nu = \{\mathbf{z} \in \mathcal{Z}^T \mid z_k > 0\},$$

the destination state function $u_{\nu,k}(\mathbf{z}) = z_k - 1$, identity otherwise, and the transition probability is $\alpha_\nu = \mu_k t_{k,0} \Delta$.

The third kind of state transitions occur when a job leaves node $k$ for node $i$, $i \neq k$. In node $k$, there must be at least one job, and in node $i$ must be buffer space available. Therefore, the source states set is

$$\mathcal{U}_\nu = \{\mathbf{z} \in \mathcal{Z}^T \mid z_k > 0,\ z_i < \nu_i\}.$$

Only the states of the nodes $k$ and $i$ change, $u_{\nu,k}(\mathbf{z}) = z_k - 1,\ u_{\nu,i}(\mathbf{z}) = z_i + 1$, and the transition probability is $\alpha_\nu = \mu_k t_{k,i} \Delta$. □

## 2 Model 1: Open tandem network, two nodes, solved with canonical aggregation

Here we consider an open tandem network with $K = 2$ nodes. Jobs arrive at node 1 according to a Poisson process, some have to wait, are served, go to node 2, have probably to wait, are served, and leave the network.

Jobs arrive with rate $\lambda_1 = 1$, are served in exponentially distributed time with rate $\mu_1 = 2$ and $\mu_2 = 3$, respectively. They go from node 1 with probability $t_{1,2} = 1$ to node 2, and leave the network from node 2 with probability $t_{2,0} = 1$.

The sum of rates with which a state is left is not greater than $\lambda_1+\mu_1+\mu_2 = 1+2+3 = 6$. Therefore we take $\Delta = 0.99/6$.

Each node has buffer space $\nu_k = 1,\ k = 1,2$, for just one job. Hence the state space of the components (nodes) is $\mathcal{S}_1 = \{0,1\}$, $\mathcal{S}_2 = \{0,1\}$, and the overall state space $\mathcal{Z}^T = \mathcal{S}_1 \times \mathcal{S}_2 = \{0,1\} \times \{0,1\}$.

Given the marginal probabilities that node $k$ is in state $j$, $r_k(j),\ j = 0,1,\ k = 1,2$, the maximum entropy distribution is $P\{\mathbf{Z} = (z_1,z_2)\} = r_1(z_1)r_2(z_2)$ ($\mathbf{Z}$ is the state random variable). Hence, disaggregation uses $\rho_{k,j} = r_k(j),\ j = 0,1,\ k = 1,2$ and $G = 1$.

At the end of [3, Sect. 1], the DA iteraton algorithm is given. It is applied now. We initialize $\rho_{k,j} = 0.5,\ j = 0,1,\ k = 1,2$ and $\epsilon = 10^{-6}$.

Aggregate transition probabilities $\pi_k(j,l) \overset{\triangle}{=} \mathrm{P}\{Z_k^{(t+1)} = l \mid Z_k^{(t)} = j\}$ ($Z_k^{(t)}$ is the state random variable of component $k$ at time $t$) are calculated using [3, (19)],

$$\pi_k(j,l) = \begin{cases} \alpha_k & \text{if } l = j-1 \geq 0, \\ 1-\beta_k & \text{if } l = j = 0, \\ 1-\alpha_k-\beta_k & \text{if } 0 < l = j < \nu_k, \\ 1-\alpha_k & \text{if } l = j = \nu_k, \\ \beta_k & \text{if } l = j+1 \leq \nu_k, \\ 0 & \text{otherwise,} \end{cases} \qquad j,l = 0,\ldots,\nu_k,$$

$$\alpha_k = \mu_k\Delta t_{k,0} + \sum_{\substack{c=1\\c\neq k}}^{K} \mu_k\Delta t_{k,c}\Big(1-r_c(\nu_c)\Big), \qquad \beta_k = \lambda_k\Delta + \sum_{\substack{c=1\\c\neq k}}^{K} \mu_c\Delta t_{c,k}\Big(1-r_c(0)\Big),$$
$$k = 1,\ldots,K.$$

From this we get:

$$\begin{aligned} \pi_1(0,1) &= \lambda_1\Delta + 0 = 0.165, \\ \pi_1(1,0) &= \mu_1\Delta 1(1-r_2(\nu_2)) = 0.330(1-r_2(\nu_2)), \\ \pi_2(0,1) &= 0 + \mu_1\Delta 1(1-r_1(0)) = 0.330(1-r_1(0)), \\ \pi_2(1,0) &= \mu_2\Delta 1 + 0 = 0.495. \end{aligned}$$

Now we present a table with figures which are calculated with the DA iteraton algorithm, in turn, namely the marginal probabilities $r_k(j),\ j = 0,1,\ k = 1,2$ with the aggregate transition equations [3, (11)],

$$r_k^{(t+1)}(l) = \sum_{j=0}^{N_k} \pi_k(j,l) r_k^{(t)}(j),\ l = 0,\ldots,N_k,\ k = 1,\ldots,C. \tag{1}$$

and the aggregate transition probabilities as just indicated:

| Step | $(r_1(0), r_1(1))$ | $\begin{bmatrix} \pi_1(0,0) & \pi_1(0,1) \\ \pi_1(1,0) & \pi_1(1,1) \end{bmatrix}$ | $(r_2(0), r_2(1))$ | $\begin{bmatrix} \pi_2(0,0) & \pi_2(0,1) \\ \pi_2(1,0) & \pi_2(1,1) \end{bmatrix}$ |
|---|---|---|---|---|
| 1 | $(0.5, 0.5)$ | $\begin{bmatrix} 0.835 & 0.165 \\ 0.165 & 0.835 \end{bmatrix}$ | $(0.5, 0.5)$ | $\begin{bmatrix} 0.835 & 0.165 \\ 0.495 & 0.505 \end{bmatrix}$ |
| 2 | $(0.5, 0.5)$ | $\begin{bmatrix} 0.835 & 0.165 \\ 0.219 & 0.781 \end{bmatrix}$ | $(0.665, 0.335)$ | $\begin{bmatrix} 0.835 & 0.165 \\ 0.495 & 0.505 \end{bmatrix}$ |
| 3 | $(0.527, 0.473)$ | $\begin{bmatrix} 0.835 & 0.165 \\ 0.238 & 0.762 \end{bmatrix}$ | $(0.721, 0.279)$ | $\begin{bmatrix} 0.844 & 0.156 \\ 0.495 & 0.505 \end{bmatrix}$ |
| ... | ... | ... | ... | ... |
| 30 | $(0.614, 0.386)$ | | $(0.795, 0.205)$ | |

Alternatively, the aggregate transition probabilities can be calculatetd using transition classes with [1, (11)] if the aggregation is canonical,

$$\pi_k(j,l) = \frac{1}{r_k(j)} \sum_{\nu=1}^{\bar{\nu}} \begin{cases} \alpha_\nu \mathrm{P}\{\mathbf{Z} \in \mathcal{U}_\nu \cap \mathcal{Z}^T \cap \mathcal{X}\} \\ \quad \text{if } j \in \mathcal{U}_{\nu,k},\ u_{\nu,k} = l, \\ 0 \text{ otherwise,} \end{cases} \tag{2}$$

where $\mathcal{X} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_{k-1} \times \{j\} \times \mathcal{S}_{k+1} \times \ldots \times \mathcal{S}_K$. We present also the calculation with transition classes because it is more general.

Our model has three transition classes: The first one, $TC_1$, accounts for the arrivals at node 1,

$$\begin{aligned} \mathcal{U}_1 &= \{0\} \times \{0,1\}, \\ u_1(y_1, y_2) &= (y_1 + 1, y_2), \\ \alpha_1 &= \lambda_1 \Delta, \end{aligned}$$

the second one, $TC_2$, accounts for the transitions of jobs from node 1 to node 2,

$$\begin{aligned} \mathcal{U}_2 &= \{(1,0)\}, \\ u_2(y_1, y_2) &= (y_1 - 1, y_2 + 1), \\ \alpha_2 &= \mu_1 \Delta, \end{aligned}$$

and the third one, $TC_3$, accounts for the departures from node 2,

$$\begin{aligned} \mathcal{U}_3 &= \{0,1\} \times \{1\}, \\ u_3(y_1, y_2) &= (y_1, y_2 - 1), \\ \alpha_3 &= \mu_2 \Delta. \end{aligned}$$

With (2) we obtain

$$\begin{aligned}
\pi_1(0,1) &= \frac{1}{r_1(0)}\lambda_1\Delta P\{\mathbf{Z}\in\{0\}\times\{0,1\}\cap\{0,1\}^2\cap\{0\}\times\{0,1\}\} \\
&= \frac{1}{r_1(0)}\lambda_1\Delta P\{\mathbf{Z}\in\{0\}\times\{0,1\}\} \\
&= \frac{1}{r_1(0)}\lambda_1\Delta r_1(0) = \lambda_1\Delta, \\
\pi_1(0,0) &= 1-\pi_1(0,1), \\
\pi_1(1,0) &= \frac{1}{r_1(1)}\mu_1\Delta P\{\mathbf{Z}\in\{(1,0)\}\cap\{0,1\}^2\cap\{1\}\times\{0,1\}\} \\
&= \frac{1}{r_1(1)}\mu_1\Delta P\{\mathbf{Z}=(1,0)\} \\
&= \frac{1}{r_1(1)}\mu_1\Delta r_1(1)r_2(0) = \mu_1\Delta r_2(0), \\
\pi_1(1,1) &= 1-\pi_1(1,0), \\
\pi_2(0,1) &= \frac{1}{r_2(0)}\mu_1\Delta P\{\mathbf{Z}\in\{(1,0)\}\cap\{0,1\}^2\cap\{0,1\}\times\{0\}\} \\
&= \frac{1}{r_2(0)}\mu_1\Delta P\{\mathbf{Z}=(0,1)\} \\
&= \frac{1}{r_2(0)}\mu_1\Delta r_1(1)r_2(0) = \mu_1\Delta r_1(1), \\
\pi_2(0,0) &= 1-\pi_2(0,1), \\
\pi_2(1,0) &= \frac{1}{r_2(1)}\mu_2\Delta P\{\mathbf{Z}\in\{0,1\}\times\{1\}\cap\{0,1\}^2\cap\{0,1\}\times\{1\}\} \\
&= \frac{1}{r_2(1)}\mu_2\Delta P\{\mathbf{Z}\in\{0,1\}\times\{1\}\} \\
&= \frac{1}{r_2(1)}\mu_2\Delta r_2(1) = \mu_2\Delta, \\
\pi_2(1,1) &= 1-\pi_2(1,0).
\end{aligned}$$

# 3 Model 2: Closed cyclic network, three nodes, solved with canonical aggregation

Here we consider a closed cyclic network with $K=3$ nodes and $N=2$ jobs. Jobs go from node 1 to node 2, from there to node 3, and back to node 1.

Jobs are served in exponentially distributed time with rate $\mu_1=1$, $\mu_2=2$, and $\mu_3=3$, respectively. They go from node 1 with probability $t_{1,2}=1$ to node 2, from node 2 with probability $t_{2,3}=1$ to node 3, and from node 3 with probability $t_{3,1}=1$ back to node 1.

The sum of rates with which a state is left is not greater than $\mu_1+\mu_2+\mu_3=1+2+3=6$. Therefore we take $\Delta=0.99/6$.

Each node has buffer space $\nu_k=1,\ k=1,2,3$, for just one job. Hence the state space of the components (nodes) is $\mathcal{S}_1 = \{0,1\}$, $\mathcal{S}_2 = \{0,1\}$, $\mathcal{S}_3 = \{0,1\}$, and the overall state space is $\mathcal{Z}^T = \{(1,1,0),(1,0,1),(0,1,1)\}$.

Given the MED parameters $\boldsymbol{\rho}$, the state probabilities are

$$p(\mathbf{z}) = \frac{1}{G}\prod_{k=1}^{K} \rho_{k,z_k},$$

[3, (2)]. The normalizing constant $G$ and the marginal probabilities are efficiently calculated with convolution [3, p. 14]:

$$G = (\boldsymbol{\rho}_1 \otimes \ldots \otimes \boldsymbol{\rho}_K)_N,$$

$$r_k(j) = \frac{\rho_{k,j}}{G}\left(\bigotimes_{\substack{c=1\\c\neq k}}^{K} \boldsymbol{\rho}_c\right)_{N-j}, \quad j = \tau_k,\ldots,\nu_k,\ k=1,\ldots,K,$$

$$r_{k,k'}(j,j') = \frac{\rho_{k,j}\rho_{k',j'}}{G}\left(\bigotimes_{\substack{c=1\\c\neq k,k'}}^{K} \boldsymbol{\rho}_c\right)_{N-j-j'}, \quad (j,j')\in \mathcal{S}_k\times\mathcal{S}_{k'},\quad \{k,k'\}\subseteq[1:K],$$

where $r_{k,k'}(j,j') \triangleq P\{Z_k = j, Z_{k'} = j'\}$, $\boldsymbol{\rho}_c \triangleq (\rho_{c,\tau_c},\ldots,\rho_{c,\nu_c})$, $c = 1,\ldots,K$, and $\otimes$ is the convolution operator ($\rho_{k,j} = 0$ for all $j \notin [\tau_k : \nu_k]$).

The MED parameters $\boldsymbol{\rho}$ are calculated approximately with [3, (14)],

$$\rho_{k,j}^{(t+1)} = \Big(1 - \omega_t + \omega_t \frac{r_k^{(t+1)}(j)}{r_k^{(t)}(j)}\Big)\rho_{k,j}^{(t)}.$$

Here we use the damping factor $\omega_i = 1$.

Our model has three transition classes: The first one, $TC_1$, accounts for the transitions of jobs from node 1 to node 2,

$$\begin{aligned}\mathcal{U}_1 &= \{1\}\times\{0\}\times\{0,1\},\\ u_1(y_1,y_2,y_3) &= (y_1-1, y_2+1, y_3),\\ \alpha_1 &= \mu_1\Delta = 0.165,\end{aligned}$$

the second one, $TC_2$, accounts for the transitions of jobs from node 2 to node 3,

$$\begin{aligned}\mathcal{U}_2 &= \{0,1\}\times\{1\}\times\{0\},\\ u_2(y_1,y_2,y_3) &= (y_1, y_2-1, y_3+1),\\ \alpha_2 &= \mu_2\Delta = 0.33,\end{aligned}$$

and the third one, $TC_3$, accounts for the transitions of jobs from node 3 to node 1,

$$\begin{aligned}\mathcal{U}_3 &= \{0\}\times\{0,1\}\times\{1\},\\ u_3(y_1,y_2,y_3) &= (y_1+1, y_2, y_3-1),\\ \alpha_3 &= \mu_3\Delta = 0.495.\end{aligned}$$

The aggregate transition probabilities are calculatetd using the transition classes with (2),

$$
\begin{aligned}
\pi_1(0,1) &= \frac{1}{r_1(0)}\alpha_3 P\{\mathbf{Z} \in \{0\} \times \{0,1\} \times \{1\} \cap \mathcal{Z}^T \cap \{0\} \times \{0,1\} \times \{0,1\}\} \\
&= \frac{1}{r_1(0)}\alpha_3 r_{1,3}(0,1), \\
\pi_1(0,0) &= 1 - \pi_1(0,1), \\
\pi_1(1,0) &= \frac{1}{r_1(1)}\alpha_1 P\{\mathbf{Z} \in \{1\} \times \{0\} \times \{0,1\} \cap \mathcal{Z}^T \cap \{1\} \times \{0,1\} \times \{0,1\}\} \\
&= \frac{1}{r_1(1)}\alpha_1 r_{1,2}(1,0), \\
\pi_1(1,1) &= 1 - \pi_1(1,0), \\
\pi_2(0,1) &= \frac{1}{r_2(0)}\alpha_1 P\{\mathbf{Z} \in \{1\} \times \{0\} \times \{0,1\} \cap \mathcal{Z}^T \cap \{0,1\} \times \{0\} \times \{0,1\}\} \\
&= \frac{1}{r_2(0)}\alpha_1 r_{1,2}(1,0), \\
\pi_2(0,0) &= 1 - \pi_2(0,1), \\
\pi_2(1,0) &= \frac{1}{r_2(1)}\alpha_2 P\{\mathbf{Z} \in \{0,1\} \times \{1\} \times \{0\} \cap \mathcal{Z}^T \cap \{0,1\} \times \{1\} \times \{0,1\}\} \\
&= \frac{1}{r_2(1)}\alpha_2 r_{2,3}(1,0), \\
\pi_2(1,1) &= 1 - \pi_2(1,0), \\
\pi_3(0,1) &= \frac{1}{r_3(0)}\alpha_2 P\{\mathbf{Z} \in \{0,1\} \times \{1\} \times \{0\} \cap \mathcal{Z}^T \cap \{0,1\} \times \{0,1\} \times \{0\}\} \\
&= \frac{1}{r_3(0)}\alpha_2 r_{2,3}(1,0), \\
\pi_3(0,0) &= 1 - \pi_3(0,1), \\
\pi_3(1,0) &= \frac{1}{r_3(1)}\alpha_3 P\{\mathbf{Z} \in \{0\} \times \{0,1\} \times \{1\} \cap \mathcal{Z}^T \cap \{0,1\} \times \{0,1\} \times \{1\}\} \\
&= \frac{1}{r_3(1)}\alpha_3 r_{1,3}(0,1), \\
\pi_3(1,1) &= 1 - \pi_3(1,0).
\end{aligned}
$$

When the DA method is applied, $C$ partitions of the state space $\mathcal{Z}^T$ are considered, in general. In partition $c$, the aggregates are defined by the state of a (composed) component which consists of one or some elementary components $h, h', \ldots \in \mathcal{H}_c,\ c = 1, \ldots, C$. In our queueing networks, the elementary components are the nodes. In the special case of canonical aggregation, $C = K$ and $\mathcal{H}_c = \{c\},\ c = 1, \ldots, C$, holds.

The DA iteraton algorithm [3, Sect. 1] is initialized $\rho_{k,j} = 1,\ j = 0, 1,\ k = 1, 2, 3$, and $\epsilon = 10^{-6}$. The marginal probabilities must be calculated twice, in step 2) with the aggregate transition equations, and in step 3) with the new MED parameters $\rho_{k,l}$.

Now we present figures which a program produced:

```
*************************** D A I ****************************
Canonical aggregation
C partitions with the following tuples; C=       3
c=       1, H_c={  1}
c=       2, H_c={  2}
c=       3, H_c={  3}
DAI, R initially:   r_1(j)
     0.33333     0.66666
   c=       2   r_c(j)
     0.33333     0.66666
   c=       3   r_c(j)
     0.33333     0.66666
Step it=       1
   Component c=       1   H_c={  1}
      Transition from aggregate j=       0, component state jj=      0
                 into aggregate l=       1, component state ll=      1
         via transition class nu=       3
         U_nu /\ Z^T /\ X =
             {  0}
            x{  0,  1}
            x{  1}
         P{U_nu /\ Z^T /\ X}=      0.33333
         Transition probability pi_c(j,l)=      0.49499
      Transition from aggregate j=       1, component state jj=      1
                 into aggregate l=       0, component state ll=      0
         via transition class nu=       1
         U_nu /\ Z^T /\ X =
             {  1}
            x{  0}
            x{  0,  1}
         P{U_nu /\ Z^T /\ X}=      0.33333
         Transition probability pi_c(j,l)=      0.08249
   Component c=       2   H_c={  2}
      Transition from aggregate j=       0, component state jj=      0
                 into aggregate l=       1, component state ll=      1
         via transition class nu=       1
         U_nu /\ Z^T /\ X =
             {  1}
            x{  0}
            x{  0,  1}
         P{U_nu /\ Z^T /\ X}=      0.33333
         Transition probability pi_c(j,l)=      0.16499
      Transition from aggregate j=       1, component state jj=      1
                 into aggregate l=       0, component state ll=      0
         via transition class nu=       2
         U_nu /\ Z^T /\ X =
             {  0,  1}
            x{  1}
```

```
          x{  0}
        P{U_nu /\ Z^T /\ X}=      0.33333
        Transition probability pi_c(j,l)=      0.16499
   Component c=       3   H_c={  3}
      Transition from aggregate j=       0, component state jj=      0
                 into aggregate l=       1, component state ll=      1
        via transition class nu=       2
        U_nu /\ Z^T /\ X =
            {  0,  1}
          x{  1}
          x{  0}
        P{U_nu /\ Z^T /\ X}=      0.33333
        Transition probability pi_c(j,l)=      0.32999
      Transition from aggregate j=       1, component state jj=      1
                 into aggregate l=       0, component state ll=      0
        via transition class nu=       3
        U_nu /\ Z^T /\ X =
            {  0}
          x{  0,  1}
          x{  1}
        P{U_nu /\ Z^T /\ X}=      0.33333
        Transition probability pi_c(j,l)=      0.24749
   New R with aggregate transition equations:
     0.22333     0.77666
     0.38833     0.61166
     0.38833     0.61166
   rho
     0.67000     1.16499
     1.16499     0.91749
     1.16499     0.91749
   R from the new rho
     0.18464     0.81535
     0.40767     0.59232
     0.40767     0.59232
Step it=       2
   Component c=       1   H_c={  1}
      Transition from aggregate j=       0, component state jj=      0
                 into aggregate l=       1, component state ll=      1
        via transition class nu=       3
        U_nu /\ Z^T /\ X =
            {  0}
          x{  0,  1}
          x{  1}
        P{U_nu /\ Z^T /\ X}=      0.18464
        Transition probability pi_c(j,l)=      0.49499
      Transition from aggregate j=       1, component state jj=      1
                 into aggregate l=       0, component state ll=      0
        via transition class nu=       1
```

```
      U_nu /\ Z^T /\ X =
          {  1}
         x{  0}
         x{  0,  1}
      P{U_nu /\ Z^T /\ X}=      0.40767
      Transition probability pi_c(j,l)=      0.08249
Component c=       2   H_c={  2}
   Transition from aggregate j=       0, component state jj=      0
              into aggregate l=       1, component state ll=      1
      via transition class nu=       1
      U_nu /\ Z^T /\ X =
          {  1}
         x{  0}
         x{  0,  1}
      P{U_nu /\ Z^T /\ X}=      0.40767
      Transition probability pi_c(j,l)=      0.16499
   Transition from aggregate j=       1, component state jj=      1
              into aggregate l=       0, component state ll=      0
      via transition class nu=       2
      U_nu /\ Z^T /\ X =
          {  0,  1}
         x{  1}
         x{  0}
      P{U_nu /\ Z^T /\ X}=      0.40767
      Transition probability pi_c(j,l)=      0.22712
Component c=       3   H_c={  3}
   Transition from aggregate j=       0, component state jj=      0
              into aggregate l=       1, component state ll=      1
      via transition class nu=       2
      U_nu /\ Z^T /\ X =
          {  0,  1}
         x{  1}
         x{  0}
      P{U_nu /\ Z^T /\ X}=      0.40767
      Transition probability pi_c(j,l)=      0.32999
   Transition from aggregate j=       1, component state jj=      1
              into aggregate l=       0, component state ll=      0
      via transition class nu=       3
      U_nu /\ Z^T /\ X =
          {  0}
         x{  0,  1}
         x{  1}
      P{U_nu /\ Z^T /\ X}=      0.18464
      Transition probability pi_c(j,l)=      0.15430
New R with aggregate transition equations:
  0.16051     0.83948
  0.47494     0.52505
  0.36454     0.63545
```

```
   rho
     0.58242     1.19948
     1.35722     0.81330
     1.04174     0.98431
   R from the new rho
     0.15114     0.84885
     0.51943     0.48056
     0.32942     0.67057
Step it=       3  ......

#DA-iterations      18

DAI aggregate probabilities R
     0.18181     0.81818
     0.54545     0.45454
     0.27272     0.72727
```

# 4 Model 3: Open tandem network, three nodes, solved with fine aggregation

Here we consider an open tandem network with $K = 3$ nodes. Jobs arrive at node 1 according to a Poisson process, some have to wait, are served, go to node 2, have probably to wait, are served, go to node 3, have probably to wait, are served, and leave the network.

Jobs arrive with rate $\lambda_1 = 1$, are served in exponentially distributed time with rate $\mu_1 = 2$, $\mu_2 = 3$, and $\mu_3 = 4$, respectively. They go from node 1 with probability $t_{1,2} = 1$ to node 2, from node 2 with probability $t_{2,3} = 1$ to node 3, and leave the network from node 3 with probability $t_{3,0} = 1$.

The sum of rates with which a state is left is not greater than $\lambda_1 + \mu_1 + \mu_2 + \mu_3 = 1 + 2 + 3 + 4 = 10$. Therefore we take $\Delta = 0.99/10$.

Each node has buffer space $\nu_k = 1,\ k = 1, 2, 3$, for just one job. Hence the state space of the components (nodes) is $\mathcal{S}_1 = \{0, 1\}$, $\mathcal{S}_2 = \{0, 1\}$, $\mathcal{S}_3 = \{0, 1\}$, and the overall state space $\mathcal{Z}^T = \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3 = \{0, 1\} \times \{0, 1\} \times \{0, 1\}$.

Here, we use the *all connected pairs* aggregation, we define the aggregates according to the state of two nodes, namely

$$\mathcal{Z}_{c,(l,l')} = \{\mathcal{I}(\mathbf{z}) \mid \mathbf{z} \in \mathcal{Z}^T,\ z_c = l,\ z_{c+1} = l'\},\ \ (l, l') \in \mathcal{S}_c \times \mathcal{S}_{c+1},\ c = 1, 2.$$

Hence, we have $C = 2$ partitions.

The aggregate probabilities are joint marginal probabilities, namely

$$r_c(j, j') \stackrel{\triangle}{=} \mathrm{P}\{Z_c = j,\ Z_{c+1} = j'\},\ j \in \mathcal{S}_c,\ j' \in \mathcal{S}_{c+1},\ c = 1, \ldots, C.$$

The state probabilities are

$$p(\mathbf{z}) = \rho_{1,z_1,z_2} \cdot \ldots \cdot \rho_{K-1,z_{K-1},z_K}/G,\ \mathbf{z} \in \mathcal{Z}^T,$$

but they are not used explicitly.

The normalization constant $G$ is efficiently calculated as follows,

$$\begin{aligned} G &= \sum_{\mathbf{z}\in\mathcal{Z}^T} \rho_{1,z_1,z_2} \cdot \ldots \cdot \rho_{K-1,z_{K-1},z_K} \\ &= \sum_{z_1\in\mathcal{S}_1} \sum_{z_2\in\mathcal{S}_2} \rho_{1,z_1,z_2} \cdot \ldots \cdot \sum_{z_K\in\mathcal{S}_K} \rho_{K-1,z_{K-1},z_K}, \end{aligned}$$

and similarly the marginal probabilities,

$$\begin{aligned} r_k(j) &= \sum_{\substack{\mathbf{z}\in\mathcal{Z}^T \\ z_k=j}} \rho_{1,z_1,z_2} \cdot \ldots \cdot \rho_{K-1,z_{K-1},z_K} \\ & \quad j\in\mathcal{S}_k,\ k=1,\ldots,K, \\ r_c(j,j') &= \sum_{\substack{\mathbf{z}\in\mathcal{Z}^T \\ z_c=j,z_{c+1}=j'}} \rho_{1,z_1,z_2} \cdot \ldots \cdot \rho_{K-1,z_{K-1},z_K} \\ & \quad j\in\mathcal{S}_c,\ j'\in\mathcal{S}_{c+1},\ c=1,\ldots,K-1. \end{aligned}$$

Our model has four transition classes: The first one, $TC_1$, accounts for the transitions of jobs from node 1 to node 2,

$$\begin{aligned} \mathcal{U}_1 &= \{1\}\times\{0\}\times\{0,1\}, \\ u_1(y_1,y_2,y_3) &= (y_1-1,y_2+1,y_3), \\ \alpha_1 &= \mu_1\Delta, \end{aligned}$$

the second one, $TC_2$, accounts for the transitions of jobs from node 2 to node 3,

$$\begin{aligned} \mathcal{U}_2 &= \{0,1\}\times\{1\}\times\{0\}, \\ u_2(y_1,y_2,y_3) &= (y_1,y_2-1,y_3+1), \\ \alpha_2 &= \mu_2\Delta, \end{aligned}$$

the third one, $TC_3$, accounts for the departures from node 3,

$$\begin{aligned} \mathcal{U}_3 &= \{0,1\}\times\{0,1\}\times\{1\}, \\ u_3(y_1,y_2,y_3) &= (y_1,y_2,y_3-1), \\ \alpha_3 &= \mu_3\Delta, \end{aligned}$$

and the fourth one, $TC_4$, accounts for the arrivals at node 1,

$$\begin{aligned} \mathcal{U}_4 &= \{0\}\times\{0,1\}\times\{0,1\}, \\ u_4(y_1,y_2,y_3) &= (y_1+1,y_2,y_3), \\ \alpha_4 &= \lambda_1\Delta. \end{aligned}$$

In the DA iteration algorithm, new macro probabilities are calculated in step 2) with the aggregate transition equations which are here

$$r_c^{(t+1)}(l,l') = \sum_{j\in\mathcal{S}_c} \sum_{j'\in\mathcal{S}_{c+1}} \pi_c\Big((j,j'),(l,l')\Big) r_c^{(t)}(j,j'),\ \ (l,l')\in\mathcal{S}_c\times\mathcal{S}_{c+1},\ c=1,\ldots,K-1.$$

For the aggregate transition probabilities, we use the general formula which is here

$$\pi_c(j,j',l,l') = \frac{1}{r_c(j,j')} \sum_{\nu=1}^{\bar{\nu}} \begin{cases} \alpha_\nu \mathrm{P}\{\mathbf{Z} \in \mathcal{U}_\nu \cap \mathcal{Z}^T \cap \mathcal{X}\} \\ \quad \text{if } j \in \mathcal{U}_{\nu,c},\ j' \in \mathcal{U}_{\nu,c+1},\ u_{\nu,c} = l,\ u_{\nu,c+1} = l', \\ 0 \text{ otherwise,} \end{cases}$$

where $\mathcal{X} = \mathcal{S}_1 \times \ldots \times \mathcal{S}_{c-1} \times \{j\} \times \{j'\} \times \mathcal{S}_{c+2} \times \ldots \times \mathcal{S}_K$.

In step 3 of the DA iteration algorithm, the MED parameters $\rho_{c,j,j'},\ (j,j') \in \mathcal{S}_c \times \mathcal{S}_{c+1},\ c = 1,\ldots,K-1$, are approximately calculated with

$$\rho^{(t+1)}_{c,j,j'} = \Big(1 - \omega_t + \omega_t \frac{r_c^{(t+1)}(j,j')}{r_c^{(t)}(j)}\Big)\rho^{(t)}_{c,j,j'}.$$

Here, too, we use the damping factor $\omega_i = 1$ (In this special case, disaggregation could be done exactly, [3, section 2.1], but we use a general program).

With these new $\boldsymbol{\rho}$, the new macro probabilities $\mathbf{R}$ are calculated.

Now we present figures which a program produced:

```
**************************** D A I *****************************
Aggregation ACP
C partitions with the following tuples; C=       2
c=       1, H_c={  1,  2}
c=       2, H_c={  2,  3}
DAI, R initially:  r_1(j)
     0.25000     0.25000     0.25000     0.25000
   c=       2   r_c(j)
     0.25000     0.25000     0.25000     0.25000
Step it=       1
   Component c=       1   H_c={  1,  2}
      Transition from aggregate j=       0, component state jj=      0       0
                 into aggregate l=       2, component state ll=      1       0
         via transition class nu=       4
         U_nu /\ Z^T /\ X =
             {  0}
            x{  0}
            x{  0,  1}
         P{U_nu /\ Z^T /\ X}=r_c(j)
         Transition probability pi_c(j,l)=      0.09899
      Transition from aggregate j=       1, component state jj=      0       1
                 into aggregate l=       0, component state ll=      0       0
         via transition class nu=       2
         U_nu /\ Z^T /\ X =
             {  0}
            x{  1}
            x{  0}
         P{U_nu /\ Z^T /\ X}=      0.12500
         Transition probability pi_c(j,l)=      0.14849
      Transition from aggregate j=       1, component state jj=      0       1
```

```
          into aggregate l=       3, component state ll=      1       1
    via transition class nu=       4
    U_nu /\ Z^T /\ X =
        {  0}
       x{  1}
       x{  0,  1}
    P{U_nu /\ Z^T /\ X}=r_c(j)
    Transition probability pi_c(j,l)=      0.09899
 Transition from aggregate j=       2, component state jj=      1       0
          into aggregate l=       1, component state ll=      0       1
    via transition class nu=       1
    U_nu /\ Z^T /\ X =
        {  1}
       x{  0}
       x{  0,  1}
    P{U_nu /\ Z^T /\ X}=r_c(j)
    Transition probability pi_c(j,l)=      0.19799
 Transition from aggregate j=       3, component state jj=      1       1
          into aggregate l=       2, component state ll=      1       0
    via transition class nu=       2
    U_nu /\ Z^T /\ X =
        {  1}
       x{  1}
       x{  0}
    P{U_nu /\ Z^T /\ X}=      0.12500
    Transition probability pi_c(j,l)=      0.14849
Component c=       2   H_c={  2,  3}
 Transition from aggregate j=       0, component state jj=      0       0
          into aggregate l=       2, component state ll=      1       0
    via transition class nu=       1
    U_nu /\ Z^T /\ X =
        {  1}
       x{  0}
       x{  0}
    P{U_nu /\ Z^T /\ X}=      0.12500
    Transition probability pi_c(j,l)=      0.09899
 Transition from aggregate j=       1, component state jj=      0       1
          into aggregate l=       3, component state ll=      1       1
    via transition class nu=       1
    U_nu /\ Z^T /\ X =
        {  1}
       x{  0}
       x{  1}
    P{U_nu /\ Z^T /\ X}=      0.12500
    Transition probability pi_c(j,l)=      0.09899
 Transition from aggregate j=       1, component state jj=      0       1
          into aggregate l=       0, component state ll=      0       0
    via transition class nu=       3
```

```
        U_nu /\ Z^T /\ X =
            {  0,  1}
           x{  0}
           x{  1}
        P{U_nu /\ Z^T /\ X}=r_c(j)
        Transition probability pi_c(j,l)=      0.39599
     Transition from aggregate j=       2, component state jj=      1       0
                into aggregate l=       1, component state ll=      0       1
        via transition class nu=       2
        U_nu /\ Z^T /\ X =
            {  0,  1}
           x{  1}
           x{  0}
        P{U_nu /\ Z^T /\ X}=r_c(j)
        Transition probability pi_c(j,l)=      0.29699
     Transition from aggregate j=       3, component state jj=      1       1
                into aggregate l=       2, component state ll=      1       0
        via transition class nu=       3
        U_nu /\ Z^T /\ X =
            {  0,  1}
           x{  1}
           x{  1}
        P{U_nu /\ Z^T /\ X}=r_c(j)
        Transition probability pi_c(j,l)=      0.39599
   New R with aggregate transition equations:
     0.26237     0.23762     0.26237     0.23762
     0.32424     0.20049     0.29949     0.17575
   rho
     1.04949     0.95049     1.04949     0.95049
     1.29699     0.80199     1.19799     0.70300
   R from the new rho
     0.27468     0.22531     0.27468     0.22531
     0.33946     0.20991     0.28397     0.16664
Step it=       2
   Component c=       1   H_c={  1,  2}
     Transition from aggregate j=       0, component state jj=      0       0
                into aggregate l=       2, component state ll=      1       0
        via transition class nu=       4
        U_nu /\ Z^T /\ X =
            {  0}
           x{  0}
           x{  0,  1}
        P{U_nu /\ Z^T /\ X}=r_c(j)
        Transition probability pi_c(j,l)=      0.09899
     Transition from aggregate j=       1, component state jj=      0       1
                into aggregate l=       0, component state ll=      0       0
        via transition class nu=       2
        U_nu /\ Z^T /\ X =
```

```
        {  0}
       x{  1}
       x{  0}
    P{U_nu /\ Z^T /\ X}=      0.14198
    Transition probability pi_c(j,l)=      0.18716
  Transition from aggregate j=       1, component state jj=      0       1
             into aggregate l=       3, component state ll=      1       1
    via transition class nu=       4
    U_nu /\ Z^T /\ X =
        {  0}
       x{  1}
       x{  0,  1}
    P{U_nu /\ Z^T /\ X}=r_c(j)
    Transition probability pi_c(j,l)=      0.09899
  Transition from aggregate j=       2, component state jj=      1       0
             into aggregate l=       1, component state ll=      0       1
    via transition class nu=       1
    U_nu /\ Z^T /\ X =
        {  1}
       x{  0}
       x{  0,  1}
    P{U_nu /\ Z^T /\ X}=r_c(j)
    Transition probability pi_c(j,l)=      0.19799
  Transition from aggregate j=       3, component state jj=      1       1
             into aggregate l=       2, component state ll=      1       0
    via transition class nu=       2
    U_nu /\ Z^T /\ X =
        {  1}
       x{  1}
       x{  0}
    P{U_nu /\ Z^T /\ X}=      0.14198
    Transition probability pi_c(j,l)=      0.18716
Component c=       2   H_c={  2,  3}
  Transition from aggregate j=       0, component state jj=      0       0
             into aggregate l=       2, component state ll=      1       0
    via transition class nu=       1
    U_nu /\ Z^T /\ X =
        {  1}
       x{  0}
       x{  0}
    P{U_nu /\ Z^T /\ X}=      0.16973
    Transition probability pi_c(j,l)=      0.09899
  Transition from aggregate j=       1, component state jj=      0       1
             into aggregate l=       3, component state ll=      1       1
    via transition class nu=       1
    U_nu /\ Z^T /\ X =
        {  1}
       x{  0}
```

```
          x{  1}
       P{U_nu /\ Z^T /\ X}=      0.10495
       Transition probability pi_c(j,l)=      0.09899
    Transition from aggregate j=       1, component state jj=      0       1
               into aggregate l=       0, component state ll=      0       0
       via transition class nu=       3
       U_nu /\ Z^T /\ X =
           {  0,  1}
          x{  0}
          x{  1}
       P{U_nu /\ Z^T /\ X}=r_c(j)
       Transition probability pi_c(j,l)=      0.39599
    Transition from aggregate j=       2, component state jj=      1       0
               into aggregate l=       1, component state ll=      0       1
       via transition class nu=       2
       U_nu /\ Z^T /\ X =
           {  0,  1}
          x{  1}
          x{  0}
       P{U_nu /\ Z^T /\ X}=r_c(j)
       Transition probability pi_c(j,l)=      0.29699
    Transition from aggregate j=       3, component state jj=      1       1
               into aggregate l=       2, component state ll=      1       0
       via transition class nu=       3
       U_nu /\ Z^T /\ X =
           {  0,  1}
          x{  1}
          x{  1}
       P{U_nu /\ Z^T /\ X}=r_c(j)
       Transition probability pi_c(j,l)=      0.39599
   New R with aggregate transition equations:
     0.28966     0.21522     0.28966     0.20544
     0.38898     0.19034     0.29923     0.12143
   rho
     1.10672     0.90794     1.10672     0.86669
     1.48618     0.72725     1.26235     0.51227
   R from the new rho
     0.30435     0.20019     0.30435     0.19109
     0.40871     0.19999     0.27833     0.11295
Step it=       3  ......

#DA-iterations      54

DAI aggregate probabilities R
     0.62399     0.37600
     0.76559     0.23440
     0.84399     0.15600
```

## References

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[3] J. Ch. Strelen, B. Bärk, J. Becker, and V. Jonas. Analysis of queueing networks with blocking using a new aggregation technique. *Annals of Operations Research*, 79:121–142, 1998.