

Likelihood Functions

The likelihood function answers the question:

What does the sensor tell about the state \mathbf{x} of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1, \rho_F = 0$

at each time one measurement:

$$p(\mathbf{z}_k | \mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k; \mathbf{H}\mathbf{x}_k, \mathbf{R})$$

- **real conditions, one object:** $P_D < 1, \rho_F > 0$

at each time n_k measurements $Z_k = \{\mathbf{z}_k^1, \dots, \mathbf{z}_k^{n_k}\}!$

$$p(Z_k, n_k | \mathbf{x}_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}(\mathbf{z}_k^j; \mathbf{H}\mathbf{x}_k, \mathbf{R})$$

Bayes Filtering for: $P_D < 1, \rho_F > 0$, well-separated objects

state \mathbf{x}_k , **current data** $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{m_k}$, **accumulated data** $\mathcal{Z}^k = \{Z_k, \mathcal{Z}^{k-1}\}$

interpretation hypotheses E_k for Z_k

object not detected, $1 - P_D$
 $\mathbf{z}_k \in Z_k$ from object, P_D } $m_k + 1$ interpretations

interpretation histories H_k for \mathcal{Z}^k

- tree structure: $H_k = (E_{H_k}, H_{k-1}) \in \mathcal{H}^k$
- current: E_{H_k} , prehistories: H_{k-i}

$$p(\mathbf{x}_k | \mathcal{Z}^k) = \sum_{H_k} p(\mathbf{x}_k, H_k | \mathcal{Z}^k) = \sum_{H_k} \underbrace{p(H_k | \mathcal{Z}^k)}_{\text{weight!}} \underbrace{p(\mathbf{x}_k | H_k, \mathcal{Z}^k)}_{\text{given } H_k: \text{ unique}} \quad \text{‘mixture’ density}$$

Closer look: $P_D < 1, \rho_F > 0$, well-separated targets

filtering (at time t_{k-1}): $p(\mathbf{x}_{k-1} | \mathcal{Z}^{k-1}) = \sum_{H_{k-1}} p_{H_{k-1}} \mathcal{N}(\mathbf{x}_{k-1}; \mathbf{x}_{H_{k-1}}, \mathbf{P}_{H_{k-1}})$

prediction (for time t_k):

$$\begin{aligned} p(\mathbf{x}_k | \mathcal{Z}^{k-1}) &= \int d\mathbf{x}_{k-1} p(\mathbf{x}_k | \mathbf{x}_{k-1}) p(\mathbf{x}_{k-1} | \mathcal{Z}^{k-1}) \quad (\text{MARKOV model}) \\ &= \sum_{H_{k-1}} p_{H_{k-1}} \mathcal{N}(\mathbf{x}_k; \mathbf{F}\mathbf{x}_{H_{k-1}}, \mathbf{F}\mathbf{P}_{H_{k-1}}\mathbf{F}^\top + \mathbf{D}) \end{aligned}$$

measurement likelihood:

$$\begin{aligned} p(Z_k, m_k | \mathbf{x}_k) &= \sum_{j=0}^{m_k} p(Z_k | E_k^j, \mathbf{x}_k, m_k) P(E_k^j | x_k, m_k) \quad (E_k^j: \text{interpretations}) \\ &\propto (1 - P_D) \rho_F + P_D \sum_{j=1}^{m_k} \mathcal{N}(z_k^j; \mathbf{H}\mathbf{x}_k, \mathbf{R}) \quad (\mathbf{H}, \mathbf{R}, P_D, \rho_F) \end{aligned}$$

filtering (at time t_k):

$$\begin{aligned} p(\mathbf{x}_k | \mathcal{Z}^k) &\propto p(Z_k, m_k | \mathbf{x}_k) p(\mathbf{x}_k | \mathcal{Z}^{k-1}) \quad (\text{BAYES' rule}) \\ &= \sum_{H_k} p_{H_k} \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{H_k}, \mathbf{P}_{H_k}) \quad (\text{Exploit product formula}) \end{aligned}$$

Exercise: Show that $p(\mathbf{x}_k | \mathcal{Z}^k)$ is given by Kalman updates and weights $p_{H_k}^j$.

$$p(\mathbf{x}_k | \mathcal{Z}^k) = \sum_{j=0}^{m_k} \sum_{H_{k-1}} p_{H_{k-1}}^j \mathcal{N}(\mathbf{x}_{k-1}; \mathbf{x}_{H_{k-1}}^j, \mathbf{P}_{H_{k-1}}^j)$$

$$p_{H_{k-1}}^j = \frac{p_{H_{k-1}}^{j*}}{\sum_j p_{H_{k-1}}^{j*}}$$

$$p_{H_{k-1}}^{j*} = p_{H_{k-1}} \begin{cases} (1 - P_D) \rho_F & j=0 \\ \frac{P_D}{\sqrt{|2\pi \mathbf{S}_{H_{k-1}}^j|}} e^{-\frac{1}{2} \boldsymbol{\nu}_{H_k}^{j\top} \mathbf{S}_{H_{k-1}}^{j-1} \boldsymbol{\nu}_{H_k}^j} & j \neq 0 \end{cases}$$

Insert mixtures and exploit product formula in the numerator and denominator!

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m data, N hypotheses $\rightarrow N^{m+1}$ continuations

radical solution: mono-hypothesis approximation

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look for *smallest statistical distance*: $\min_i \|\nu_{k|k-1}^i\|$

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\rightarrow **Nearest-Neighbor filter (NN)**

+ one hypothesis, - hard decision, - not adaptive

- **global combining:** Merge all hypotheses!

\rightarrow **PDAF, JPDAF filter**

+ all data, + adaptive, - reduced applicability

PDAF Filter: formally analog to Kalman Filter

Filtering (scan $k-1$): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_{k-1}; \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})$ (\rightarrow initiation)

prediction (scan k): $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$ (like Kalman)

Filtering (scan k): $p(\mathbf{x}_k|\mathcal{Z}^k) \approx \sum_{j=0}^{m_k} p_k^j \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{k|k}^j, \mathbf{P}_{k|k}^j) \approx \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

$$\boldsymbol{\nu}_k = \sum_{j=0}^{m_k} p_k^j \boldsymbol{\nu}_k^j, \quad \boldsymbol{\nu}_k^j = \mathbf{z}_k^j - \mathbf{H}\mathbf{x}_{k|k-1} \quad \text{combined innovation}$$

$$\mathbf{W}_k = \mathbf{P}_{k|k-1}\mathbf{H}^\top\mathbf{S}_k^{-1}, \quad \mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}_k \quad \text{Kalman gain matrix}$$

$$p_k^j = p_k^{j*} / \sum_j p_k^{j*}, \quad p_k^{j*} = \begin{cases} (1 - P_D) \rho_F \\ \frac{P_D}{\sqrt{|2\pi\mathbf{S}_{H_k}|}} e^{-\frac{1}{2}\boldsymbol{\nu}_{H_k}^\top \mathbf{S}_{H_k} \boldsymbol{\nu}_{H_k}} \end{cases} \quad \text{weighting factors}$$

$$\mathbf{x}_k = \mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k \quad \text{(Filtering Update: Kalman)}$$

$$\mathbf{P}_k = \mathbf{P}_{k|k-1} - (1 - p_k^0) \mathbf{W}_k \mathbf{S} \mathbf{W}_k^\top \quad \text{(Kalman part)}$$

$$+ \mathbf{W}_k \left\{ \sum_{j=0}^{m_k} p_k^j \boldsymbol{\nu}_k^j \boldsymbol{\nu}_k^{j\top} - \boldsymbol{\nu}_k \boldsymbol{\nu}_k^\top \right\} \mathbf{W}_k^\top \quad \text{(Spread of Innovations)}$$

The qualitative shape of $p(\mathbf{x}_k | \mathcal{Z}^k)$ is often much simpler than its correct representation: *a few pronounced modes*

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before continuing existing track hypotheses H_{k-1}
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after calculating the weights p_{H_k} , before filtering
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after calculating the weights p_{H_k} , before filtering
→ *limiting case:* Nearest Neighbor filter (NN)
- **local combining:** Merge *similar hypotheses!*
after the complete calculation of the pdfs
→ *limiting case:* PDAF (global combining)

Successive Local Combining

Partial sums of *similar* densities \rightarrow moment matching:

$$\sum_{H_k \in \mathcal{H}^{k*}} p_{H_k} \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{H_k}, \mathbf{P}_{H_k}) \approx p_{H_k^*} \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{H_k^*}, \mathbf{P}_{H_k^*})$$

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similarity: $d(H_1, H_2) < \mu$ mit (z.B.):

$$d(H_1, H_2) = (\mathbf{x}_{H_1} - \mathbf{x}_{H_2})^\top (\mathbf{P}_{H_1} + \mathbf{P}_{H_2})^{-1} (\mathbf{x}_{H_1} - \mathbf{x}_{H_2})$$

Start: Hypothesis of highest weight $H_1 \rightarrow$ search similar hypothesis

$(p_H \searrow) \rightarrow$ merge: $(H_1, H) \succ H_1^* \rightarrow$ continue search $(p_H \searrow) \dots$

\rightarrow **restart:** hypothesis with next to highest weight $H_2 \rightarrow \dots$

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- In many cases: good approximations \rightarrow *quasi-optimality*
- PDAF, JPDAF: $\mathcal{H}^{k*} = \mathcal{H}^k \rightarrow$ *limited applicability*
- robustness \rightarrow *detail mostly irrelevant*

Retrodiction for GAUSSIAN Mixtures

wanted: $p(\mathbf{x}_l | \mathcal{Z}^k) \longleftarrow p(\mathbf{x}_{l+1} | \mathcal{Z}^k)$ for $l < k$

$$p(\mathbf{x}_l | \mathcal{Z}^k) = \sum_{H_k} p(\mathbf{x}_l, H_k | \mathcal{Z}_k) = \sum_{H_k} \underbrace{p(\mathbf{x}_l | H_k, \mathcal{Z}^k)}_{\text{no ambiguities!}} \underbrace{p(H_k | \mathcal{Z}^k)}_{\text{filtering!}}$$

Calculation of $p(\mathbf{x}_l | H_k, \mathcal{Z}^k)$ as in case of $P_D = 1, \rho_F = 0!$

$$p(\mathbf{x}_l | H_k, \mathcal{Z}^k) = \mathcal{N}(\mathbf{x}_l; \mathbf{x}_{H_k}(l|k), \mathbf{P}_{H_k}(l|k))$$

with parameters given by RAUCH-TUNG-STRIEBEL formulae:

$$\mathbf{x}_{H_k}(l|k) = \mathbf{x}_{H_k}(l|l) + \mathbf{W}_{H_k}(l|k) (\mathbf{x}_{H_k}(l+1|k) - \mathbf{x}_{H_k}(l+1|l))$$

$$\mathbf{P}_{H_k}(l|k) = \mathbf{P}_{H_k}(l|l) + \mathbf{W}_{H_k}(l|k) (\mathbf{P}_{H_k}(l+1|k) - \mathbf{P}_{H_k}(l+1|l)) \mathbf{W}_{H_k}(l|k)^\top$$

$$\text{gain matrix: } \mathbf{W}_{H_k}(l|k) = \mathbf{P}_{H_k}(l|l) \mathbf{F}_{l+1|l}^\top \mathbf{P}_{H_k}(l+1|l)^{-1}$$

Retrodiction of Hypotheses' Weights

Consider approximation: neglect RTS step!

$$p(\mathbf{x}_l | H_k, \mathcal{Z}^k) = \mathcal{N}(\mathbf{x}_l; \mathbf{x}_{H_k}(l|k), \mathbf{P}_{H_k}(l|k)) \approx \mathcal{N}(\mathbf{x}_l; \mathbf{x}_{H_k}(l|l), \mathbf{P}_{H_k}(l|l))$$

$$p(\mathbf{x}_l | H_k, \mathcal{Z}^k) \approx \sum_{H_l} p_{H_l}^* \mathcal{N}(\mathbf{x}_l; \mathbf{x}_{H_k}(l|l), \mathbf{P}_{H_k}(l|l))$$

with recursively defined weights:

$$p_{H_k}^* = p_{H_k}, \quad p_{H_l}^* = \sum p_{H_{l+1}}^*$$

summation over all histories H_{l+1} with equal pre-histories!

- Strong sons strengthen weak fathers.
- Weak sons weaken even strong fathers.
- If all sons die, also the father must die.

Track Extraction: Initiation of the PDF Iteration

extraction of target tracks: detection on a higher level of abstraction

start: data sets $Z_k = \{z_k^j\}_{j=1}^{m_k}$ (sensor performance: P_D, ρ_F, \mathbf{R})

goal: Detect a target trajectory in a time series: $\mathcal{Z}^k = \{Z_i\}_{i=1}^k!$

at first simplifying assumptions:

- The targets in the sensors' field of view (FoV) are well-separated.
- The sensor data in the FoV in scan i are produced simultaneously.

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decision between two competing hypotheses:

h_1 : Besides false returns \mathcal{Z}^k contains also target measurements.

h_0 : There is no target existing in the FoV; all data in \mathcal{Z}^k are false.

statistical decision errors:

$P_1 = \text{Prob}(\text{accept } h_1 | h_1)$ analogous to the sensors' P_D

$P_0 = \text{Prob}(\text{accept } h_1 | h_0)$ analogous to the sensors' P_F

Practical Approach: Sequential Likelihood Ratio Test

Goal: Decide as fast as possible for given decision errors $P_0, P_1!$

Consider the ratio of the conditional probabilities $p(h_1|\mathcal{Z}^k), p(h_0|\mathcal{Z}^k)$ and the likelihood ratio $LR(k) = p(\mathcal{Z}^k|h_1)/p(\mathcal{Z}^k|h_0)$ as an intuitive decision function:

$$\frac{p(h_1|\mathcal{Z}^k)}{p(h_0|\mathcal{Z}^k)} = \frac{p(\mathcal{Z}^k|h_1)}{p(\mathcal{Z}^k|h_0)} \frac{p(h_1)}{p(h_0)} \quad \text{a priori: } p(h_1) = p(h_0)$$

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Starting from a time window with length $k = 1$, calculate the test function $LR(k)$ successively and compare it with *two* thresholds A, B :

If $LR(k) < A$, accept hypothesis h_0 (i.e. no target is existing)!

If $LR(k) > B$, accept hypothesis h_1 (i.e. target exists in FoV)!

If $A < LR(k) < B$, wait for new data Z_{k+1} , repeat with $LR(k + 1)$!

Sequential LR Test: Some Useful Properties

1. Thresholds and decision errors are approximately related to each other by:

$$A \approx \frac{1 - P_1}{1 - P_0} \quad \text{and} \quad B \approx \frac{P_1}{P_0}$$

2. The *actual decision length* (number of scans required) is a random variable.
3. On average, the test has a *minimal decision length* for given errors P_0 , P_1 .
4. The quantity P_0 (P_1) affects the *mean decision length* given h_1 (h_0) holds.
5. Choose the probability P_1 close to 1 for actually detecting *real target tracks*.
6. P_0 should be *small* for not overloading the tracking system with *false tracks*.

Iterative Calculation of the Likelihood Ratio

$$LR(k) = \frac{p(\mathcal{Z}^k|h_1)}{p(\mathcal{Z}^k|h_0)} = \frac{\int d\mathbf{x}_k p(Z_k, m_k, \mathbf{x}_k, \mathcal{Z}^{k-1}|h_1)}{p(Z_k, m_k, \mathcal{Z}^{k-1}, h_0)}$$

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basic idea: iterative calculation!

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basic idea: iterative calculation!

Let $H_k = \{E_k, H_{k-1}\}$ be an interpretation history of the time series $\mathcal{Z}^k = \{Z_k, \mathcal{Z}^{k-1}\}$.

$E_k = E_k^0$: target was not detected, $E_k = E_k^j$: $\mathbf{z}_k^j \in Z_k$ is a target measurement.

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}, h_1) = \sum_{H_{k-1}} p(\mathbf{x}_k|H_{k-1}\mathcal{Z}^{k-1}, h_1) p(H_{k-1}|\mathcal{Z}^{k-1}, h_1) \quad \text{The standard MHT prediction!}$$

$$p(Z_k, m_k|\mathbf{x}_k, h_1, h_1) = \sum_{E_k} p(Z_k, E_k|\mathbf{x}_k, h_1) \quad \text{The standard MHT likelihood function!}$$

The calculation of the likelihood ratio is just a by-product of Bayesian MHT tracking.

Iteration Formula for $LR(k) = p(\mathcal{Z}^k|h_1)/p(\mathcal{Z}^k|h_0)$

initiation: $k = 0, \quad j_0 = 0, \quad \lambda_{j_0} = 1$

recursion: $LR(k+1) = \sum_{j_{k+1}} \lambda_{j_{k+1}} = \sum_{j_{k+1}=0}^{m_{k+1}} \sum_{j_k} \lambda_{j_{k+1}j_k} \lambda_{j_k}$

with: $\lambda_{j_{k+1}j_k} = \begin{cases} 1 - P_D & \text{for } j_{k+1} = 0 \\ \frac{P_D}{\rho_F} \mathcal{N}(\nu_{j_{k+1}j_k}, \mathbf{S}_{j_{k+1}j_k}) & \text{for } j_{k+1} \neq 0 \end{cases}$

convenient notation: with $\mathbf{j}_k = (j_k, \dots, j_1)$ let $\sum_{\mathbf{j}_k} \lambda_{\mathbf{j}_k} = \sum_{j_k=0}^{m_k} \dots \sum_{j_1=0}^{m_1} \lambda_{j_k \dots j_1}$

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innovation: $\nu_{j_{k+1}j_k} = \mathbf{z}_{j_{k+1}} - \mathbf{H}_{j_{k+1}} \mathbf{x}_{j_{k+1}|k}$

innov. cov.: $\mathbf{S}_{j_{k+1}j_k} = \mathbf{H}_{j_{k+1}} \mathbf{P}_{j_{k+1}|k} \mathbf{H}_{j_{k+1}}^\top + \mathbf{R}_{j_{k+1}}$

state update: $\mathbf{x}_{j_{k+1}|k} = \mathbf{F}_{j_{k+1}} \mathbf{x}_{j_k} \quad \mathbf{x}_{j_k} = \mathbf{x}_{j_{k|k-1}} + \mathbf{W}_{j_k j_{k-1}} \nu_{j_k j_{k-1}}$

covariances: $\mathbf{P}_{j_{k+1}|k} = \mathbf{F}_{j_{k+1}} \mathbf{P}_{j_k} \mathbf{F}_{j_{k+1}}^\top + \mathbf{D}_{j_{k+1}} \quad \mathbf{P}_{j_k} = \mathbf{P}_{j_{k|k-1}} - \mathbf{W}_{j_k j_{k-1}} \mathbf{S}_{j_k j_{k-1}} \mathbf{W}_{j_k j_{k-1}}^\top$

Iteration Formula for $LR(k) = p(\mathcal{Z}^k|h_1)/p(\mathcal{Z}^k|h_0)$

initiation: $k = 0, \quad j_0 = 0, \quad \lambda_{j_0} = 1$

recursion: $LR(k+1) = \sum_{j_{k+1}} \lambda_{j_{k+1}} = \sum_{j_{k+1}=0}^{m_{k+1}} \sum_{j_k} \lambda_{j_{k+1}j_k} \lambda_{j_k}$

with: $\lambda_{j_{k+1}j_k} = \begin{cases} 1 - P_D & \text{for } j_{k+1} = 0 \\ \frac{P_D}{\rho_F} \mathcal{N}(\nu_{j_{k+1}j_k}, \mathbf{S}_{j_{k+1}j_k}) & \text{for } j_{k+1} \neq 0 \end{cases}$

innovation: $\nu_{j_{k+1}j_k} = \mathbf{z}_{j_{k+1}} - \mathbf{H}_{j_{k+1}} \mathbf{x}_{j_{k+1}|k}$

innov. cov.: $\mathbf{S}_{j_{k+1}j_k} = \mathbf{H}_{j_{k+1}} \mathbf{P}_{j_{k+1}|k} \mathbf{H}_{j_{k+1}}^\top + \mathbf{R}_{j_{k+1}}$

state update: $\mathbf{x}_{j_{k+1}|k} = \mathbf{F}_{j_{k+1}} \mathbf{x}_{j_k} \quad \mathbf{x}_{j_k} = \mathbf{x}_{j_{k|k-1}} + \mathbf{W}_{j_k j_{k-1}} \nu_{j_k j_{k-1}}$

covariances: $\mathbf{P}_{j_{k+1}|k} = \mathbf{F}_{j_{k+1}} \mathbf{P}_{j_k} \mathbf{F}_{j_{k+1}}^\top + \mathbf{D}_{j_{k+1}} \quad \mathbf{P}_{j_k} = \mathbf{P}_{j_{k|k-1}} - \mathbf{W}_{j_k j_{k-1}} \mathbf{S}_{j_k j_{k-1}} \mathbf{W}_{j_k j_{k-1}}^\top$

Exercise 9.1 Show that this recursion formulae for calculating the decision function is true.

Sequential Track Extraction: Discussion

- $LR(k)$ is given by a growing number of summands, each related to a particular interpretation history. The tuple $\{\lambda_{j_k}, \mathbf{x}_{j_k}, \mathbf{P}_{j_k}\}$ is called a *sub-track*.

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- For mitigating growing memory problems all approximations discussed for track maintenance can be used if they do not significantly affect $LR(k)$:
 - *individual gating*: Exclude data not likely to be associated.
 - *pruning*: Kill sub-tacks contributing marginally to the test function.
 - *local combining*: Merge similar sub tracks:

$$\{\lambda_i, \mathbf{x}_i, \mathbf{P}_i\}_i \rightarrow \{\lambda, \mathbf{x}, \mathbf{P}\} \quad \text{with: } \lambda = \sum_i \lambda_i,$$
$$\mathbf{x} = \frac{1}{\lambda} \sum_i \lambda_i \mathbf{x}_i, \quad \mathbf{P} = \frac{1}{\lambda} \sum_i \lambda_i [\mathbf{P}_i + (\mathbf{x}_i - \mathbf{x})(\dots)^\top].$$

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- The LR test ends with a decision in favor of or against the hypotheses: h_0 (no target) or h_1 (target existing). Intuitive interpretation of the thresholds!

track extraction at t_k : Decide in favor of h_1 !

initiation of pdf iteration (track maintenance):

Normalize coefficients λ_{j_k} : $p_{j_k} = \frac{\lambda_{j_k}}{\sum_{j_k} \lambda_{j_k}}$

$$(\lambda_{j_k}, \mathbf{x}_{j_k}, \mathbf{P}_{j_k}) \rightarrow p(\mathbf{x}_k | \mathcal{Z}^k) = \sum_{j_k} p_{j_k} \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{j_k}, \mathbf{P}_{j_k})$$

Continue track extraction with the remaining sensor data!

sequential LR test for track monitoring:

After deciding in favor of h_1 reset $\text{LR}(0) = 1$! Calculate $\text{LR}(k)$ from $p(\mathbf{x}_k | \mathcal{Z}^k)$!

track confirmation: $\text{LR}(k) > \frac{P_1}{P_0}$: reset $\text{LR}(0) = 1$!

track deletion: $\text{LR}(k) < \frac{1-P_1}{1-P_0}$; ev. track re-initiation

DEMONSTRATION (simulated)

DEMONSTRATION (simulated)

Exercise 9.2 (voluntary)

Simulate a detection process with a given P_D , target measurements with a given \mathbb{R} , a detection process with a given P_D and realize the track extraction procedure.



ABRAHAM WALD (1902-1950) Austro-Hungarian mathematician who contributed to decision theory, geometry, and econometrics; founded the theory of economic equilibria in Oskar Morgenstern's institut in Vienna: "Berechnung der Ausschaltung von Saisonschwankungen" (Springer Verlag, 1936) the basis of Game Theory: Morgenstern, John von Neumann, John Forbes Nash (1994: Nobel price with Reinhard Selten, Bonn University) → sensor management! Founder of statistical sequential analysis in WW II. 1950 plenary talk at the International Congress of Mathematicians ICM, Cambridge (Mass.): "Basic ideas of a general theory of statistical decision rules" (1900: Hilbert's 23 Problems).

Student and friend: Jacob Wolfowitz (statistician, information theory), classical text book: "Coding Theorems of Information Theory" (1978). Posthumous attack by Ronald Fisher: "an incompetent book on statistics", passionately defended by Jerzy Neyman as imminent a statistician as Fisher.