Recapitulation

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0}, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

Recapitulation: A popular model for object evolutions

Piecewise Constant White Acceleration Model

$$p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)$$

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\, \mathbf{I} & \frac{1}{2}\Delta T^3\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \Delta T_k^2\, \mathbf{I} \end{pmatrix}$$

with $\Delta T_k = t_k - t_{k-1}$. Reasonable choice: $\frac{1}{2}v_{\mathsf{max}} \leq \Sigma_k \leq q_{\mathsf{max}}$

Recapitulation: A popular model for object evolutions

Piecewise Constant White Acceleration Model

$$p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)$$

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$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

Consider state vectors $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$ (position, velocity, acceleration)

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\mathbf{I} & \Delta T_k\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^2\,\mathbf{I} & \Delta T_k\mathbf{I} & \mathbf{I} \end{pmatrix}$$

with $\Delta T_k = t_k - t_{k-1}$. Reasonable choice: $\frac{1}{2}v_{\text{max}}/v_{\text{max}} \leq \Sigma_k \leq v_{\text{max}}/q_{\text{max}}$

Object evolution: Gauss-Markov process

- **linear evolution equation:** $\mathbf{x}_k = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{o}, \mathbf{D}_{k|k-1})$

Object evolution: Gauss-Markov process

- **linear evolution equation:** $\mathbf{x}_k = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{o}, \mathbf{D}_{k|k-1})$

$\mathbf{x}_k$ is an affine transformation of a Gaussian RV $\mathbf{v}_k$ with the pdf

$p(\mathbf{v}_k) = \mathcal{N}\big(\mathbf{v}_k;\, \mathbf{o},\, \mathbf{D}_{k|k-1}\big)$. Thus also $\mathbf{x}_k$ is a Gaussian RV with:

$$\mathbb{E}[\mathbf{x}_k] = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{I}\mathbf{o} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1}$$

$$\mathbb{C}[\mathbf{x}_k] = \mathbf{I}\mathbf{D}_{k|k-1}\mathbf{I}^\top = \mathbf{D}_{k|k-1}$$

Recapitulation: Affine Transforms of GAUSSian RVs

$$\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{D}\big)$ with $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D}\to 0$$

$$= \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top+\mathbf{D}\big) \quad \text{for } \mathbf{D}\to 0; \quad \text{product formula!}$$

Also true if $\dim(\mathbf{x}) \neq \dim(\mathbf{y})$**!**

Object evolution: Gauss-Markov process

- **linear evolution equation:** $\mathbf{x}_k = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{o}, \mathbf{D}_{k|k-1})$

- **Very simple example:**

  Object on a strait line: 2D state $\mathbf{x}_k = (x_k, \dot{x}_k)^\top$

  simple approach:
  $$\begin{aligned} x_k &= x_{k-1} + \Delta t\,\dot{x}_{k-1} & \Delta t = t_k - t_{k-1} \\ \dot{x}_k &= \dot{x}_{k-1} + v & v \sim N(0, D) \end{aligned}$$

  we thus have: $\mathbf{x}_k = \left(\begin{smallmatrix} 1 & \Delta t \\ 0 & 1 \end{smallmatrix}\right) \mathbf{x}_{k-1} + \mathbf{v}, \quad \mathbf{v} \sim N(0, \mathbf{D}),\ \mathbf{D} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right)$

Object evolution: Gauss-Markov process

- **linear evolution equation:** $\mathbf{x}_k = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{o}, \mathbf{D}_{k|k-1})$

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- **Requested: Markov property!**

$$p\big(\mathbf{x}_k \big|\, \mathbf{x}_{k-1}, \mathbf{x}_{k-2}, \ldots, \mathbf{x}_1\big) \overset{!}{=} p\big(\mathbf{x}_k \big|\, \mathbf{x}_{k-1}\big) \overset{!}{=} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}_{k|k-1}\big)$$

Fraunhofer FKIE

Object evolution: Gauss-Markov process

- **linear evolution equation:** $\mathbf{x}_k = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{o}, \mathbf{D}_{k|k-1})$

- **Very simple example:**

  Object on a strait line: 2D state $\mathbf{x}_k = (x_k, \dot{x}_k)^\top$

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  we thus have: $\mathbf{x}_k = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \mathbf{x}_{k-1} + \mathbf{v}, \quad \mathbf{v} \sim N(0, \mathbf{D}), \; \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$

- **Requested: Markov property!**

$$p(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{x}_{k-2}, \ldots, \mathbf{x}_1) \stackrel{!}{=} p(\mathbf{x}_k | \mathbf{x}_{k-1}) \stackrel{!}{=} \mathcal{N}(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1})$$

therefore: $$\begin{aligned} p(\mathbf{x}_k) &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k, \mathbf{x}_{k-1}, \ldots \mathbf{x}_1) \\ &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}, \ldots \mathbf{x}_1) \\ &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k|\mathbf{x}_{k-1}) \ldots p(\mathbf{x}_2|\mathbf{x}_1)\, p(\mathbf{x}_1) \end{aligned}$$

- **Another, rather realistic model (van Keuk):**

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} & \frac{1}{2}(t_k - t_{k-1})^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-(t_k - t_{k-1})/\theta}\,\mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \mathrm{diag}[1, 1, 1]$$

$$\mathbf{D}_{k|k-1} = \Sigma^2 (1 - \mathrm{e}^{-2(t_k - t_{k-1})/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{O} = \mathrm{diag}[0, 0, 0]$$

maneuver correlation time θ (z.B. 60 s), *limiting acceleration* Σ (z.B. 2 g)

There are many different evolution models adapted to particular problems!

- **Another, rather realistic model (van Keuk):**

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} & \frac{1}{2}(t_k - t_{k-1})^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-(t_k - t_{k-1})/\theta}\,\mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \mathrm{diag}[1,1,1]$$

$$\mathbf{D}_{k|k-1} = \Sigma^2(1 - \mathrm{e}^{-2(t_k - t_{k-1})/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{O} = \mathrm{diag}[0,0,0]$$

maneuver correlation time θ (z.B. 60 s), *limiting acceleration* Σ (z.B. 2 g)

There are many different evolution models adapted to particular problems!

Show for the acceleration process:

Exercise 5.1 (voluntary!)

$$\mathbb{E}[\ddot{\mathbf{r}}_k] = 0, \quad \mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top] = \Sigma^2\,\mathrm{e}^{-(t_k - t_l)/\theta}\,\mathbf{I},\ l \leq k$$

$\mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top]$ is called ‘auto correlation function’.

Fraunhofer FKIE

Filtering Step: An Alternative Formulation

$$p(\mathbf{x}_k|\,\mathcal{Z}^k) = p(\mathbf{x}_k|\,\mathbf{z}_k, \mathcal{Z}^{k-1}) \qquad \text{(current measurement)}$$

$$= \frac{p(\mathbf{z}_k|\,\mathbf{x}_k)\, p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, p(\mathbf{z}_k|\,\mathbf{x}_k)\, p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})} \qquad \text{(Bayes' rule)}$$

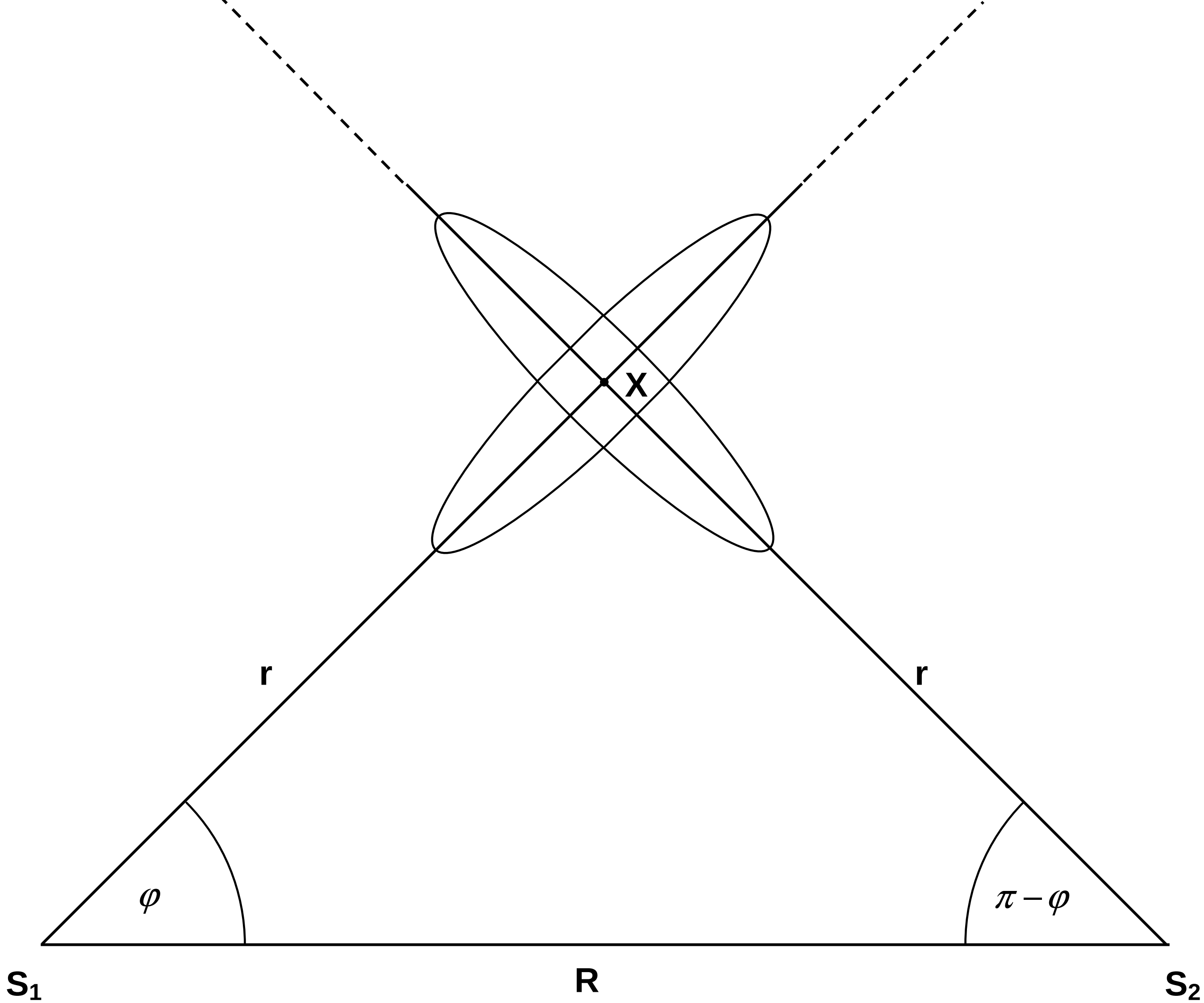
$$= \frac{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)\; \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}{\int d\mathbf{x}_k\, \underbrace{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)}_{\text{likelihood function}}\; \underbrace{\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}_{\text{prediction for } t_k}}$$

$$= \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}) \qquad \text{(product formula: 2. version!)}$$

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\left(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{z}_k\right)$$

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{H}$$

inverse covariance matrices are called information matrices.


X
r
r
φ
$\pi - \varphi$
S_1
R
S_2

Special case: *stationary* object

Example: different sensors

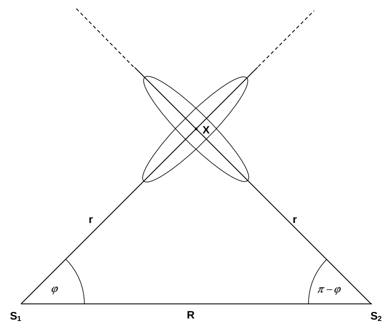
$$\mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$$
$$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k \quad \textit{time dependent!}$$

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Filtering: $\mathbf{x}_{k|k} = \mathbf{P}_{k|k} \sum_{i=1}^{k} \mathbf{R}_i^{-1} \mathbf{z}_i, \qquad \mathbf{P}_{k|k} = \Big(\sum_{i=1}^{k} \mathbf{R}_i^{-1} \Big)^{-1}$

Kalman filter $\rightarrow$ *weighted*, recursive, arithmetic mean

estimation error covariance matrix: harmonic mean of measurement error matrices!



Fraunhofer FKIE

Discussion: stationary objects

- If all measurement error covariances R_i, $i = 1, \ldots, k$ are identical, we observe the statistical “square-root effect”: $P_{k|k} = R/k$
- If the corresponding error ellipses are significantly different in their geometric extension, we can observe a much larger effect.
- statistical “intersection” of error ellipses: *harmonic mean*!
- In the limiting case of very eccentric error ellipses, we obtain triangulation of a position from bearings ($\rightarrow$ multiple sensor data fusion!).
- These considerations are valid also for 3D and more abstract measurements. The corresponding intersections: not intuitively clear.

Sensor data: range, azimuth, range-rate

Coordinates: Sensor data $\to$ *polar*, object evolution $\to$ *Cartesian*

Dynamics system: $\mathbf{x}^d = (x, y, \dot{x}, \dot{y})$

Sensor system: $\mathbf{x}^s = (r, \varphi, \dot{r}, \dot{\varphi})$

	scan $k-1$		scan k		scan k
Dynamics system	$p\big(\mathbf{x}^d_{k-1} \mid \mathcal{Z}^{k-1}\big)$	$\xrightarrow[\text{Dynamics}]{p(\mathbf{x}^d_k \mid \mathbf{x}^d_{k-1})}$	$p\big(\mathbf{x}^d_k \mid \mathcal{Z}^{k-1}\big)$		$p\big(\mathbf{x}^d_k \mid \mathcal{Z}^k\big)$
	$\uparrow \mathbf{t}_{d\leftarrow s}$		$\downarrow \mathbf{t}_{s\leftarrow d}$		$\uparrow \mathbf{t}_{d\leftarrow s}$
Sensor system	$p\big(\mathbf{x}^s_{k-1} \mid \mathcal{Z}^{k-1}\big)$		$p\big(\mathbf{x}^s_k \mid \mathcal{Z}^{k-1}\big)$	$\xrightarrow[\text{Sensor}]{p(\mathbf{z}_k \mid \mathbf{x}^s_k)}$	$p\big(\mathbf{x}^s_k \mid \mathcal{Z}^k\big)$

non-linear **coordinate transformations:**

$$\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s] = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{pmatrix} \qquad \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d] = \begin{pmatrix} r \\ \varphi \\ \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan y/x \\ (x\dot{x}+y\dot{y})/\sqrt{x^2+y^2} \\ (x\dot{y}-y\dot{x})/(x^2+y^2) \end{pmatrix}$$

Extended *Kalman* filter: be wise - linearize!

non-lineare transformations: Taylor expansion up to 1st order

around $\mathbf{x}^s_{k|k}$ (filtering): $\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_k] \approx \mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]\,(\mathbf{x}^s_k - \mathbf{x}^s_{k|k})$

mit: $\mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]{=}\partial\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]/\partial\mathbf{x}^s_{k|k}$ (Jacobian)

around $\mathbf{x}^d_{k|k-1}$ (Prediction): $\mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_k] \approx \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_{k|k-1}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}]\,(\mathbf{x}^d_k - \mathbf{x}^d_{k|k-1})$

with: $\mathbf{T}_{s\leftarrow d}{=}\partial\mathbf{t}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}]/\partial\mathbf{x}^d_{k|k-1}$

affine transformation of Gaussian random variables:

$$\mathcal{N}\big(x;\, \mathrm{x},\, \mathbf{X}\big) \xrightarrow{y=\mathbf{a}+\mathbf{A}x} \mathcal{N}\big(y;\, \mathbf{a}+\mathbf{A}\mathrm{x},\, \mathbf{A}\mathbf{X}\mathbf{A}^\top\big)$$

Exercise 5.2 (voluntary!) Calculate Jacobians $\mathbf{T}_{d\leftarrow s}$ and $\mathbf{T}_{s\leftarrow d}$.

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)$$

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$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big)\ \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

Fraunhofer FKIE

Retrodiction: How to calculate the pdf $p(\mathrm{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

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$$p\big(\mathrm{x}_l|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l, \mathrm{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big)\; \underbrace{p\big(\mathrm{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

$$p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big) = \frac{p(Z_k, \ldots, Z_{l+1}|\mathrm{x}_{l+1}, \mathrm{x}_l, \mathcal{Z}^l)\; p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)}{\int d\mathrm{x}_l\; p(Z_k, \ldots, Z_{l+1}|\mathrm{x}_{l+1}, \mathrm{x}_l, \mathcal{Z}^l)\; p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)} = p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

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$$p(\mathbf{x}_l|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k)\; \underbrace{p(\mathbf{x}_{l+1}|\,\mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

$$p(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k) = p(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l) = \frac{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)\; p(\mathbf{x}_l|\,\mathcal{Z}^l)}{\int d\mathbf{x}_l\; \underbrace{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)}_{\text{dynamics model}}\; \underbrace{p(\mathbf{x}_l|\,\mathcal{Z}^l)}_{\text{filtering } t_l}}$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, \frac{p\big(\mathbf{x}_{l+1}|\,\mathbf{x}_l\big)\, p\big(\mathbf{x}_l|\,\mathcal{Z}^l\big)}{\int d\mathbf{x}_l\, \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathbf{x}_l\big)}_{\text{dynamics model}}\, \underbrace{p\big(\mathbf{x}_l|\,\mathcal{Z}^l\big)}_{\text{filtering } t_l}}\, \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

- $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ retrodiction: last iteration step
- $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ dynamic object behavior
- $p(\mathbf{x}_l|\mathcal{Z}^l)$ filtering at the time considered
- GAUSSians, GAUSSian mixtures: Exploit product formula!
- linear GAUSSian likelihood/dynamics: Rauch-Tung-Striebel smoothing

Exercise 5.3 Derive the *Rauch-Tung-Striebel* formulae

by using the Kalman filter assumptions

and the product formula (twice)!

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$

$$\begin{aligned}\mathbf{x}_{l|k} &= \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}),\\ \mathbf{P}_{l|k} &= \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^{\top}\end{aligned} \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^{\top}\mathbf{P}_{l+1|l}^{-1}$$

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$

Exercise 5.4 Implement the *Rauch-Tung-Striebel* formulae in your simulator!

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big) \end{aligned}$$

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big) \end{aligned}$$

Looks like a Kalman filtering update!

Fraunhofer FKIE

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}_{l+\theta|l+1}^{\top} \\
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^{\top}\mathbf{P}_{l+1|l}^{-1} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^{\top} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)$$
$$= \mathcal{N}\big(\mathbf{b}_{l+\theta|l+1};\, \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} \\
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}^\top_{l+\theta|l+1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big)$$
$$= \mathcal{N}\Big(\mathbf{b}_{l+\theta|l+1};\, \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} \\
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}^\top_{l+\theta|l+1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{x}_{l+\theta|k},\, \mathbf{x}_{l+\theta|k}\Big)$$

$$\begin{aligned}
\mathbf{x}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\left(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}\right) \\
\mathbf{P}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\left(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l}\right)\mathbf{\Phi}^\top_{l+\theta|l+1}
\end{aligned}$$

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction:

$$\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering:

$$\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

retrodiction:

$$\mathcal{N}(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k})$$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$