UNIVERSITÄT BONN

Rheinische Friedrich-Wilhelms Universität

Mathematisch-Naturwissenschaftliche Fakultät

Zur militärischen Nutzung der Künstlichen Intelligenz: Ethische, völkerrechtliche und technische Probleme

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dies academicus 2019

Universität Bonn

15. Mai 2019, 16.15 Uhr, Hörsaal I, Hauptgebäude

Längst durchzieht Künstliche Intelligenz KI alle Bereiche der Technosphäre, die uns umgibt. Immer mehr gilt dies auch für die Verteidigung und Sicherheit. Welche Risiken ergeben sich daraus? Bleiben sie nicht nur technisch, sondern auch ethisch und völkerrechtlich beherrschbar?

Nur eines der Beispiele ist *Future Combat Air System* FCAS, das europäische Luftkampfsystem der Zukunft, auf dessen Entwicklung sich Deutschland und Frankreich 2017 verständigt haben. Anders als bisherige Systeme ist FCAS weit mehr als nur ein Kampfflugzeug, sondern ein komplexes System der Systeme, das nur durch Algorithmen kontrollierbar sein wird.

Wie kaum ein Großvorhaben zuvor wirft die Integration unbemannter Subsysteme in FCAS die Frage auf, wie ethische und völkerrechtliche Regeln technisch zu implementieren sind. Vor allem ist die Vorgabe zu erfüllen, das „ein Waffeneinsatz von unbemannten Luftfahrzeugen ausschließlich unter der Kontrolle des Menschen erfolgt“, wie die „Militärische Luftfahrtstrategie“ der Bundesregierung festschreibt. Denn *being killed by a machine is the ultimate human indignity*. Zudem gilt der Koalitionsvertrag: „Autonome Waffensysteme, die der Verfügung des Menschen entzogen sind, lehnen wir ab“.

Die Diskussion ethischer, rechtlicher und ingenieurwissenschaftlicher Probleme, die nicht nur die militärische Nutzung der KI aufwirft, muss in einen gesamtgesellschaftlichen Diskurs eingebettet sein, zu dem die Vorlesung beitragen möchte. Er wird uns noch lange begleiten.

More Precise Formulation of the BAYESian Approach

Consider a set of measurements $Z_l = \{\mathbf{z}_l^j\}_{j=1}^{m_l}$ of a single or a multiple target state $\mathbf{x}_l$ at time instants t_l, $l = 1, \ldots, k$ and the time series:

$$\mathcal{Z}^k = \{Z_k, m_k, Z_{k-1}, m_{k-1}, \ldots, Z_1, m_1\} = \{Z_k, m_k, \mathcal{Z}^{k-1}\}!$$

Based on $\mathcal{Z}^k$, what can be learned about the object states $\mathbf{x}_l$ at $t_1, \ldots, t_k, t_{k+1}, \ldots$, i.e. for the past, present, and future?

Evidently the answer is given be calculating the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$!

multiple sensor measurement fusion: Calculate $p(\mathbf{x}|\mathcal{Z}_1^k, \ldots, \mathcal{Z}_N^k)$!

- communication lines
- common coordinate system: sensor registration

How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider at first the present time: $l = k$.

an observation:

Bayes' rule: $$p(\mathbf{x}_k|\mathcal{Z}^k) = p(\mathbf{x}_k|Z_k, m_k, \mathcal{Z}^{k-1})$$

$$= \frac{p(Z_k, m_k|\mathbf{x}_k, \mathcal{Z}^{k-1})\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, p(Z_k, m_k|\mathbf{x}_k, \mathcal{Z}^{k-1})\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider at first the present time: $l = k$.

an observation:

Bayes' rule: $p(\mathbf{x}_k|\mathcal{Z}^k) = p(\mathbf{x}_k|Z_k, m_k, \mathcal{Z}^{k-1})$

$$= \frac{p(Z_k, m_k|\mathbf{x}_k, \mathcal{Z}^{k-1})\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\ \underbrace{p(Z_k, m_k|\mathbf{x}_k, \mathcal{Z}^{k-1})}_{\text{likelihood function}}\ \underbrace{p(\mathbf{x}_k|\mathcal{Z}^{k-1})}_{\text{prediction}}}$$

- $p(\mathbf{x}_k|\mathcal{Z}^{k-1})$ is a *prediction* of the target state at time t_k based on all measurements in the *past*.

- $p(Z_k, m_k|\mathbf{x}_k) \propto \ell(\mathbf{x}_k; Z_k, m_k)$ describes, what the *current* sensor output Z_k, m_k can say about the current target state $\mathbf{x}_k$ and is called *likelihood function*.

Fraunhofer FKIE

- $p(\mathbf{x}_k|\mathcal{Z}^{k-1})$ is a *prediction* for time t_k based on all measurements in the *past*.

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, \mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \qquad \text{marginal pdf}$$

$$= \int d\mathbf{x}_{k-1}\, \underbrace{p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1})}_{\text{object dynamics!}}\, \underbrace{p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}_{\text{idea: iteration!}} \qquad \text{notion of a conditional pdf}$$

sometimes: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}}_{\text{deterministic}}\mathbf{x}_{k-1},\, \underbrace{\mathbf{D}_{k|k-1}}_{\text{random}}\big)$ (linear GAUSS-MARKOV)

- $p(Z_k, m_k|\mathbf{x}_k) \propto \ell(\mathbf{x}_k; Z_k, m_k)$ describes, what the *current* sensor output Z_k, m_k can say about the current target state $\mathbf{x}_k$ and is called *likelihood function*.

sometimes: $\ell(\mathbf{x}_k; \mathbf{z}_k) = \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$ (1 target, 1 measurement)

iteration formula: $$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; \mathbf{z}_k)\, \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; \mathbf{z}_k)\, \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}$$

A popular model for object evolutions

Piecewise Constant White Acceleration Model

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

For known $\mathbf{x}_{k-1}$ and without external influences we have with $\Delta T_k = t_k - t_{k-1}$:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},$$ **see blackboard!**

Assume during the interval ΔT_k a constant acceleration $\mathbf{a}_k$ causing the state evolution:

$$\begin{pmatrix} \frac{1}{2}\Delta T_k^2\, \mathbf{I} \\ \Delta T_k\, \mathbf{I} \end{pmatrix} \mathbf{a}_k =: \mathbf{G}_k \mathbf{a}_k,$$ **linear transform!**

Let $\mathbf{a}_k$ be a Gaussian RV with pdf: $p(\mathbf{a}_k) = \mathcal{N}\big(\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2 \mathbf{I}\big)$, we therefore have:

$$p(\mathbf{G}_k\mathbf{a}_k) = \mathcal{N}\big(\mathbf{G}_k\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2 \mathbf{G}_k\mathbf{G}_k^\top\big).$$

Remember: Affine Transforms of GAUSSian RVs.

$$\boxed{\mathcal{N}\big(\mathbf{x};\, \bar{\mathbf{x}},\, \mathbf{P}\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\, \mathbf{t}+\mathbf{T}\bar{\mathbf{x}},\, \mathbf{T}\mathbf{P}\mathbf{T}^\top\big)}$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x}) = \mathcal{N}\big(\mathbf{y};\, \mathbf{t}+\mathbf{T}\mathbf{x},\, \mathbf{R}\big)$ with $\mathbf{R} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y}-\mathbf{t};\mathbf{T}\mathbf{x},\mathbf{R})\, \mathcal{N}(\mathbf{x};\bar{\mathbf{x}},\mathbf{P}) \quad \text{for } \mathbf{R} \to 0$$

$$= \mathcal{N}\big(\mathbf{y};\, \mathbf{t}+\mathbf{T}\bar{\mathbf{x}},\, \mathbf{T}\mathbf{P}\mathbf{T}^\top + \mathbf{R}\big) \qquad \text{for } \mathbf{R} \to 0; \quad \text{product formula!}$$

Also true if $\dim(\mathbf{x}) \neq \dim(\mathbf{y})$**!**

Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\Big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\Big)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

Fraunhofer FKIE

Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\left(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\right)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\, \mathbf{I} & \frac{1}{2}\Delta T_k^3\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \Delta T_k^2\, \mathbf{I} \end{pmatrix}$$

Exercise 4.1 Consider $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$ (position, velocity, acceleration)

Show that $\mathbf{F}_{k|k-1}$ and $\mathbf{D}_{k|k-1} = \Sigma_k^2 \mathbf{G}_k \mathbf{G}_k^\top$ (constant acceleration rates) are given by:

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\, \mathbf{I} & \frac{1}{2}\Delta T_k^2\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\, \mathbf{I} & \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \frac{1}{2}\Delta T_k^2\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \Delta T_k^2 \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^2\, \mathbf{I} & \Delta T_k \mathbf{I} & \mathbf{I} \end{pmatrix}$$

with $\Delta T_k = t_k - t_{k-1}$. Reasonable choice: $\frac{1}{2}q_{\text{max}} \le \Sigma_k \le q_{\text{max}}$

A more Insightful Look at a Data Fusion Algorithm

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0}, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\ \mathbf{x}_{0|0},\ \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}_{k-1|k-1},\ \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\ \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\ \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k},\ \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$
'KALMAN gain matrix'

Exercise 4.2

In your sensor simulator, chose a sensor at position $\mathbf{r}_s$, for example $\mathbf{r}_s = (0, 0)^\top$, that produces measurements $\mathbf{z}_k$ of the Cartesian target positions $\mathbf{H}\mathbf{x}_k$ from your ground truth generator. Use the measurement covariance matrix $\mathbf{R} = \sigma_c^2\text{diag}[1, 1]$, σ_c = 50 m. Program your first Kalman filter using a constant acceleration or the van Keuk model. Visualize your results nicely! Compare the ground truth, the measurement, and the estimates!

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) = p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \qquad \text{independent sensors}$$

$$= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big)$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$\begin{aligned} p(\mathbf{z}_k^1, \mathbf{z}_k^2|\mathbf{x}_k) &= p(\mathbf{z}_k^1|\mathbf{x}_k)\, p(\mathbf{z}_k^2|\mathbf{x}_k) \qquad \text{independent sensors} \\ &= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \end{aligned}$$

A Useful Product Formula for GAUSSians

$$\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\, \mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \dots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \dots, S_k$:

$$\begin{aligned}
p(\mathbf{z}_k^1, \mathbf{z}_k^2|\mathbf{x}_k) &= p(\mathbf{z}_k^1|\mathbf{x}_k)\, p(\mathbf{z}_k^2|\mathbf{x}_k) \qquad \text{independent sensors} \\
&= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \\
&= \mathcal{N}\big(\mathbf{H}_k\mathbf{x}_k;\, \mathbf{z}_k^1,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \\
&\propto \mathcal{N}\big(\mathbf{H}_k\mathbf{x}_k;\, \underbrace{\mathbf{R}_k(\mathbf{R}_k^{1\,-1}\mathbf{z}_k^1 + \mathbf{R}_k^{2\,-1}\mathbf{z}_k^2)}_{=\mathbf{z}_k},\, \underbrace{(\mathbf{R}_k^{1\,-1} + \mathbf{R}_k^{2\,-1})^{-1}}_{=\mathbf{R}_k}\big) \\
&= \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)
\end{aligned}$$

Exercise 4.3 Generalize to the case $S_k > 2$ (induction argument)!

One possible fusion strategy:

Create a singe effective measurement
by preprocessing of individual sensor measurement!

$$\mathbf{z}_k = \mathbf{R}_k \sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1} \mathbf{z}_k^s$$ weighted arithmetic mean of measurements

$$\mathbf{R}_k = \left(\sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1} \right)^{-1}$$ harmonic mean of measuremt covariances

A typical structure for fusion equations!

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'Kalman gain matrix'}$$

Exercise 4.4

In your sensor simulator, chose an **arbitrary number** S **of sensors** at positions $\mathbf{r}_s$, $s = 1, \dots, S$, produce measurements $\mathbf{z}_k^s$, $s = 1, \dots, S$, of the Cartesian target positions $\mathbf{H}\mathbf{x}_k$ from your ground truth generator. Use preprocessing both algorithms! Discuss pros & cons!

Towards real world sensors: range, azimuth data

- **Gaussian measurements in polar coordinates:**

$$\mathbf{z}_k^p = (r_k, \varphi_k)^\top, \quad \text{error covariance matrix: } \mathbf{R}^p = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **transformation into Cartesian target positions:**

$$\mathbf{z}_k^c = \mathbf{t}[\mathbf{z}_k^p] = r_k \begin{pmatrix} \cos \varphi_k \\ \sin \varphi_k \end{pmatrix} \qquad \text{A non-affin transformation!}$$

A Taylor-series approximation of $\mathbf{t}[\mathbf{z}_k^p]$ up to the first order were affin!

Towards real world sensors: range, azimuth data

- **Gaussian measurements in polar coordinates:**

$$\mathbf{z}_k^p = (r_k, \varphi_k)^\top, \quad \text{error covariance matrix: } \mathbf{R}^p = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \ r, \varphi \text{ independent}$$

- **Taylor-approximation around:** $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:

$$\mathbf{z}_k^c = \mathbf{t}[\mathbf{z}_k^p] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

Towards real world sensors: range, azimuth data

- **Gaussian measurements in polar coordinates:**

$$\mathbf{z}_k^p = (r_k, \varphi_k)^\top, \quad \text{error covariance matrix: } \mathbf{R}^p = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **Taylor-approximation around:** $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:

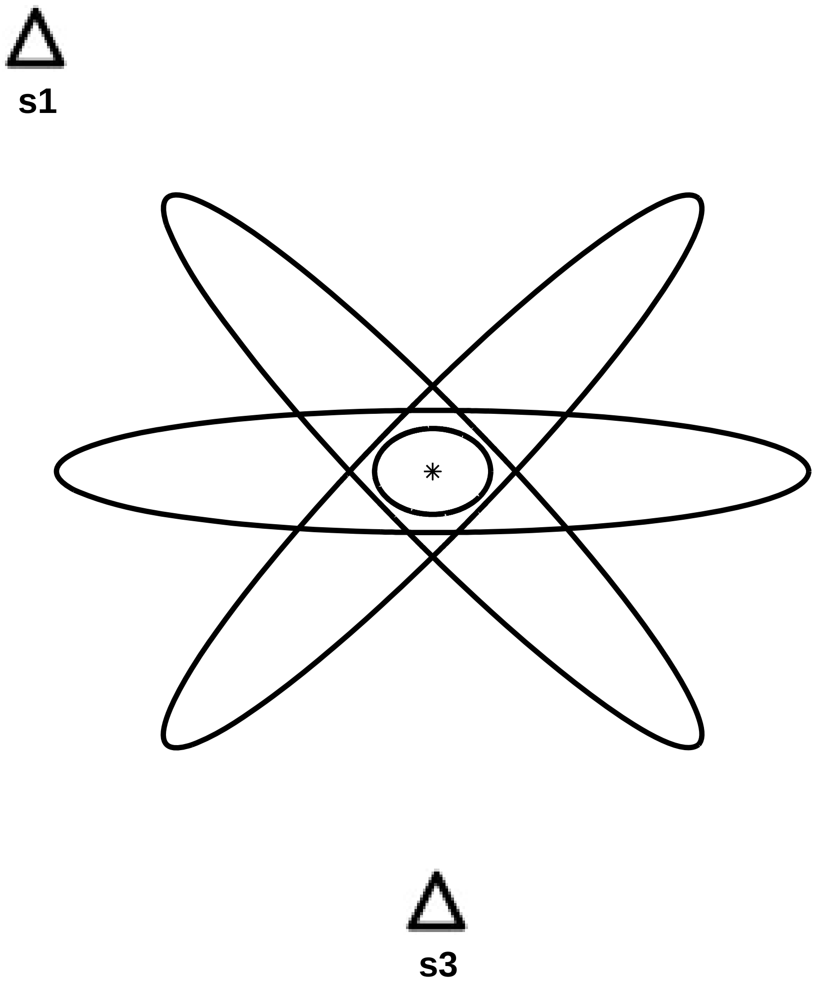
$$\mathbf{z}_k^c = \mathbf{t}[\mathbf{z}_k^p] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **Cartesian error covariance (time dependent):**

$$\boxed{\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\,\mathbf{R}\,\mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top}$$

- **sensor fusion: sensor-to-target-geometry enters into** $\mathbf{TRT}^\top$



Multiple sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$.

Typical of radar, sonar, laser scanner (lidar), cameras, microphones, ...

Fraunhofer FKIE

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

Exercise 4.5

Do the same as in exercise 5.4, but use sensors that are producing range and azimuth measurements of the target positions.