Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'KALMAN gain matrix'}$$

Fraunhofer FKIE

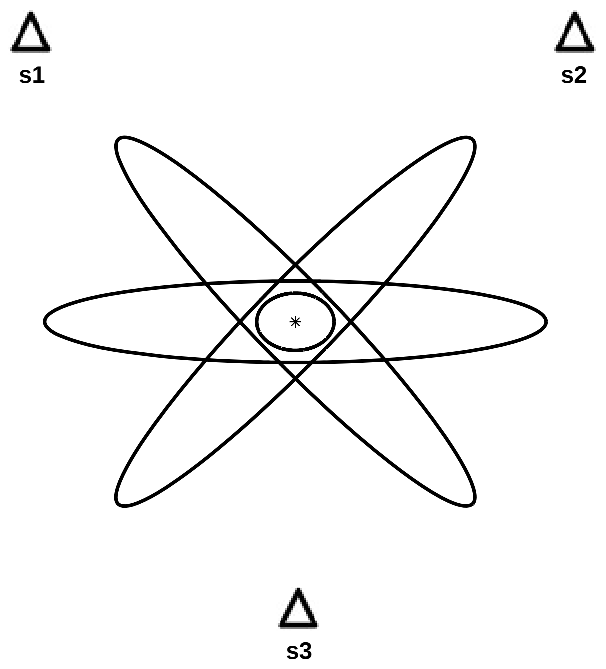
Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

a closer look: The error of each measurement $\mathbf{z}_i$ is described by a related measurement error *covariance matrix* $\mathbf{R}_i$ ('error ellipsoids'). In 2 dimensions:



$\mathbf{R}_i$ can strongly depend on the underlying senor-to-target geometry!

Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \quad r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

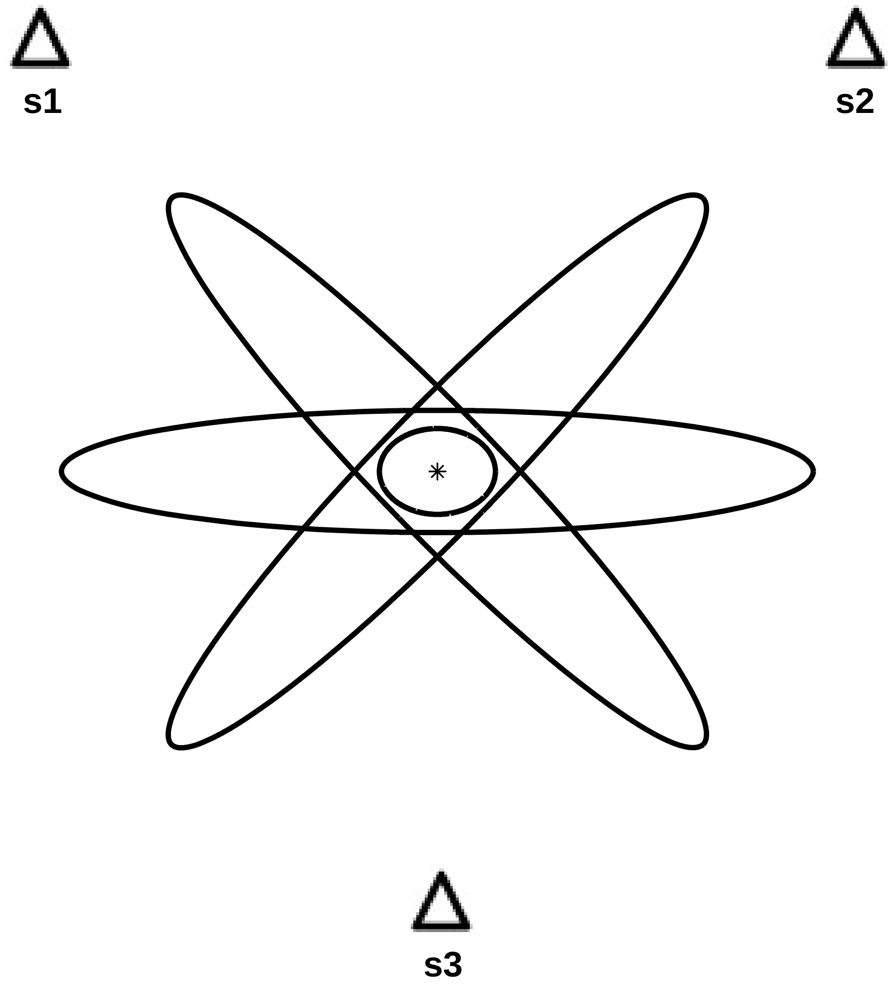
$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **Cartesian error covariance (time dependent):**

$$\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\, \mathbf{R}\, \mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top$$

- **sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$**



sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$

Filtering step: alternative formulation

$$p(\mathbf{x}_k | \mathcal{Z}^k) = p(\mathbf{x}_k | \mathbf{z}_k, \mathcal{Z}^{k-1}) \qquad \text{(current measurement)}$$

$$= \frac{p(\mathbf{z}_k | \mathbf{x}_k)\, p(\mathbf{x}_k | \mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, p(\mathbf{z}_k | \mathbf{x}_k)\, p(\mathbf{x}_k | \mathcal{Z}^{k-1})} \qquad \text{(BAYES' rule)}$$

$$= \frac{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)\; \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}{\int d\mathbf{x}_k\, \underbrace{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)}_{\text{likelihood function}}\; \underbrace{\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}_{\text{prediction for } t_k}}$$

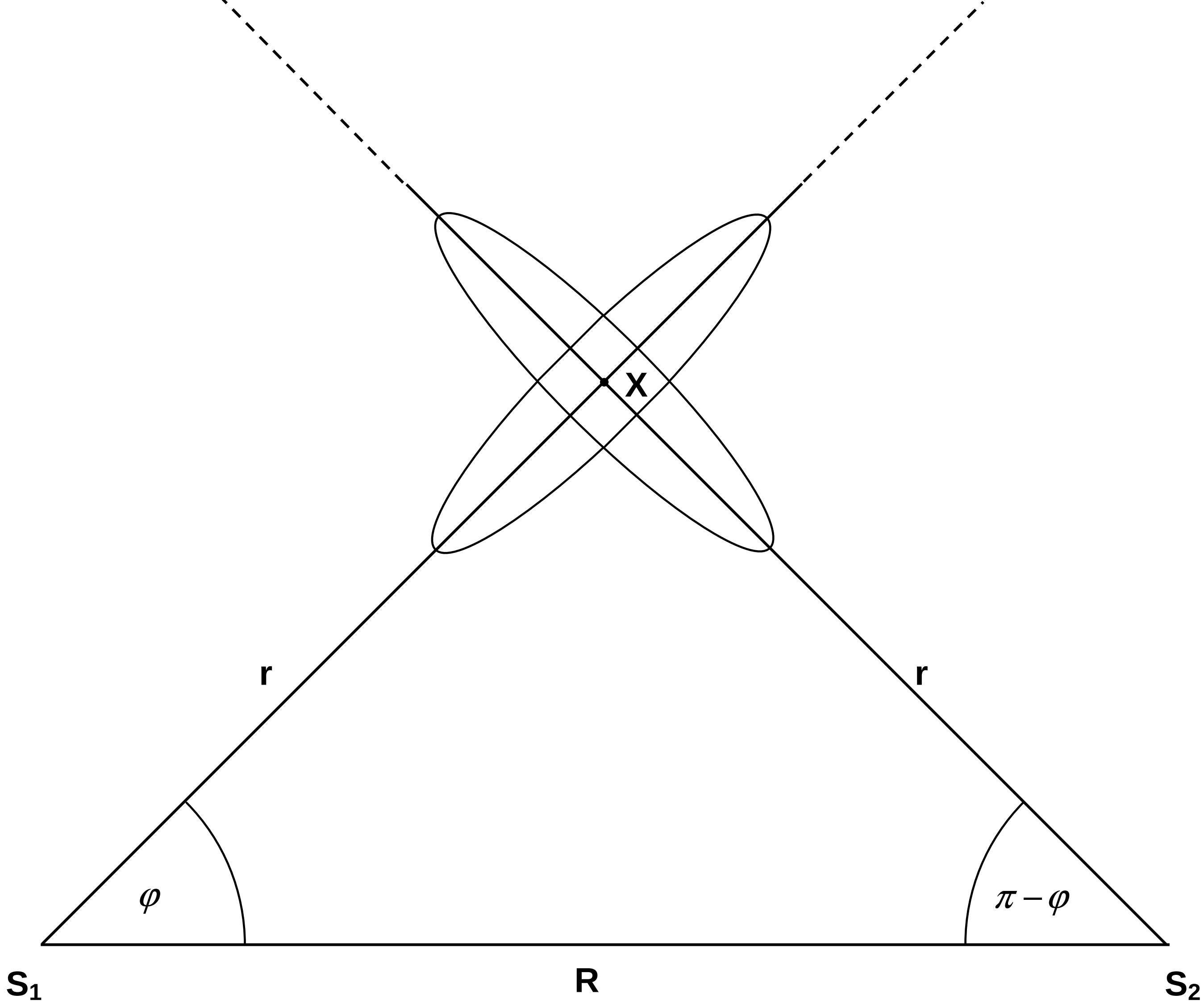
$$= \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}) \qquad \text{(product formula: 2. version!)}$$

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\left(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{z}_k\right)$$

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{H}$$

inverse covariance matrices are called information matrices.

Fraunhofer FKIE



X
r
r
φ
π − φ
S1
R
S2

Special case: *stationary* object

Example: different sensors $\quad \mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$

$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k$ ***time dependent!***

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Prediction: $\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}, \quad \mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}^{\top} + \mathbf{D}_{k|k-1}$

Filtering: $\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\big(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^{\top}\mathbf{R}_k^{-1}\mathbf{z}_k\big)$ (2. formulation)

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^{\top}\mathbf{R}_k^{-1}\mathbf{H}$$

Special case: *stationary* object

Example: different sensors $\quad \mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$
$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k$ ***time dependent!***

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Prediction: $\mathbf{x}_{k|k-1} = \mathbf{x}_{k-1|k-1}, \quad \mathbf{P}_{k|k-1} = \mathbf{P}_{k-1|k-1}$

Filtering:

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\left(\mathbf{P}_{k-1|k-1}^{-1}\mathbf{x}_{k-1|k-1} + \mathbf{R}_k^{-1}\mathbf{z}_k\right)$$

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k-1|k-1}^{-1} + \mathbf{R}_k^{-1}$$

Special case: *stationary* object

Example: different sensors

$$\mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$$
$$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k$$ ***time dependent!***

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Prediction: $\mathbf{x}_{k|k-1} = \mathbf{x}_{k-1|k-1}, \quad \mathbf{P}_{k|k-1} = \mathbf{P}_{k-1|k-1}$

Filtering:

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\big(\mathbf{P}_{k-1|k-1}^{-1}\mathbf{x}_{k-1|k-1} + \mathbf{R}_k^{-1}\mathbf{z}_k\big) = \mathbf{P}_{k|k}^{-1}\sum_{i=1}^{k}\mathbf{R}_i^{-1}\mathbf{z}_i$$

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k-1|k-1}^{-1} + \mathbf{R}_k^{-1} = \sum_{i=1}^{k}\mathbf{R}_i^{-1}$$

Fraunhofer FKIE

Special case: *stationary* object

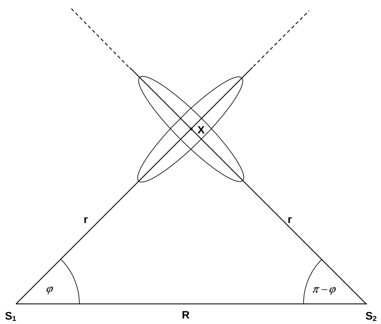
Example: different sensors $\mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$
$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k$ ***time dependent!***

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Filtering: $\mathbf{x}_{k|k} = \mathbf{P}_{k|k} \sum_{i=1}^{k} \mathbf{R}_i^{-1} \mathbf{z}_i, \qquad \mathbf{P}_{k|k} = \left(\sum_{i=1}^{k} \mathbf{R}_i^{-1} \right)^{-1}$

Kalman filter $\rightarrow$ *weighted*, recursive, arithmetic mean

estimation error covariance matrix: harmonic mean of measurement error matrices!



Fraunhofer
FKIE

Discussion: stationary objects

- If all measurement error covariances R_i, $i = 1, \ldots, k$ are identical, we observe the statistical "square-root effect": $P_{k|k} = R/k$

- If the corresponding error ellipses are significantly different in their geometric extension, we can observe a much larger effect.

- statistical "intersection" of error ellipses: *harmonic mean*!

- In the limiting case of very eccentric error ellipses, we obtain triangulation of a position from bearings ($\rightarrow$ multiple sensor data fusion!).

- These considerations are valid also for 3D and more abstract measurements. The corresponding intersections: not intuitively clear.