

$$\begin{aligned}
& \mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P}) \\
&= \underbrace{\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{y}, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x}; \mathbf{y} + \mathbf{W}\boldsymbol{\nu}, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x}; \mathbf{Q}(\mathbf{P}^{-1}\mathbf{y} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}), \mathbf{Q}) \end{cases} \\
&\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1} \\
&\mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.
\end{aligned}$$

- Interpret  $\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P})$  as a joint density:  $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x}) p(\mathbf{x})!$ 
  - Show that  $p(\mathbf{z}, \mathbf{x})$  is a Gaussian:  $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\left(\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix}; \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}\right)!$
  - Calculate from  $p(\mathbf{z}, \mathbf{x})$  the marginal and conditional densities  $p(\mathbf{z})$ ,  $p(\mathbf{x}|\mathbf{z})!$

$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \quad \text{and} \quad p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z}, \mathbf{x})}{p(\mathbf{z})}$$

- Therefore:  $p(\mathbf{z}|\mathbf{x}) p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z}) p(\mathbf{z})!$

$$p(\mathbf{z}, \mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underbrace{[(\mathbf{z} - \mathbf{H}\mathbf{x})^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{y})^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{y})]}_{=:(*)} \right\}$$

**Strategy:** Try to rewrite (\*) in such a way, that the state vector  $\mathbf{u} = (\mathbf{z}, \mathbf{x})^\top$  appears in a quadratic form  $(\mathbf{u} - \mathbf{v})^\top \mathbf{U}^{-1}(\mathbf{u} - \mathbf{v})$ .

**Do  $\mathbf{u}$ ,  $\mathbf{z}$  or  $\mathbf{x}$  not appear (\*) elsewhere, we have shown that  $\mathbf{u}$  is Gaussian distributed with expectation value  $\mathbf{v}$  and covariance matrix  $\mathbf{U}$ . For all multiplicative constants result from knowing that  $p(\mathbf{u}) = \mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P})$  is a pdf, i.e. correctly normalized.**

$$p(\mathbf{z}, \mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underbrace{[(\mathbf{z} - \mathbf{H}\mathbf{x})^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{y})^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{y})]}_{=:(*)} \right\}$$

**Strategy:** Try to rewrite (\*) in such a way, that the state vector  $\mathbf{u} = (\mathbf{z}, \mathbf{x})^\top$  appears in a quadratic form  $(\mathbf{u} - \mathbf{v})^\top \mathbf{U}^{-1}(\mathbf{u} - \mathbf{v})$ .

Do  $\mathbf{u}$ ,  $\mathbf{z}$  or  $\mathbf{x}$  not appear (\*) elsewhere, we have shown that  $\mathbf{u}$  is Gaussian distributed with expectation value  $\mathbf{v}$  and covariance matrix  $\mathbf{U}$ . For all multiplicative constants result from knowing that  $p(\mathbf{u}) = \mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P})$  is a pdf, i.e. correctly normalized.

This can be obtained by a completion of the squares, i.e. by using one of the binomial formulae (vector version, proof: multiply!):

$$(\mathbf{u} - \mathbf{v})^\top \mathbf{U}^{-1}(\mathbf{u} - \mathbf{v}) = \mathbf{u}^\top \mathbf{U}^{-1}\mathbf{u} - 2\mathbf{u}^\top \mathbf{U}^{-1}\mathbf{v} + \mathbf{v}^\top \mathbf{U}^{-1}\mathbf{v}$$

$$p(\mathbf{z}, \mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underbrace{[(\mathbf{z} - \mathbf{H}\mathbf{x})^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{y})^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{y})]}_{=:(*)} \right\}$$

auxilliary quantities:  $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$(*) = (\mathbf{A}\mathbf{u})^\top \mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + (\mathbf{B}(\mathbf{u} - \mathbf{y}'))^\top \mathbf{P}^{-1}(\mathbf{B}(\mathbf{u} - \mathbf{y}'))$$

$$p(\mathbf{z}, \mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underbrace{[(\mathbf{z} - \mathbf{H}\mathbf{x})^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{y})^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{y})]}_{=:(*)} \right\}$$

auxilliary quantities:  $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned} (*) &= (\mathbf{A}\mathbf{u})^\top \mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + (\mathbf{B}(\mathbf{u} - \mathbf{y}'))^\top \mathbf{P}^{-1}(\mathbf{B}(\mathbf{u} - \mathbf{y}')) \\ &= \mathbf{u}^\top (\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A}) \mathbf{u} + (\mathbf{u} - \mathbf{y}')^\top (\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}) (\mathbf{u} - \mathbf{y}') \quad \text{multiply!} \end{aligned}$$

$$p(\mathbf{z}, \mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underbrace{[(\mathbf{z} - \mathbf{H}\mathbf{x})^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{y})^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{y})]}_{=:(*)} \right\}$$

auxiliary quantities:  $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned} (*) &= (\mathbf{A}\mathbf{u})^\top \mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + (\mathbf{B}(\mathbf{u} - \mathbf{y}'))^\top \mathbf{P}^{-1}(\mathbf{B}(\mathbf{u} - \mathbf{y}')) \\ &= \mathbf{u}^\top (\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A}) \mathbf{u} + (\mathbf{u} - \mathbf{y}')^\top (\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}) (\mathbf{u} - \mathbf{y}') \quad \text{multiply!} \\ &= \mathbf{u}^\top \underbrace{(\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A} + \mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B})}_{=:\mathbf{U}^{-1}} \mathbf{u} - 2\mathbf{u}^\top \underbrace{(\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}) \mathbf{y}'}_{=:\mathbf{U}^{-1}\mathbf{v}} + \text{const.} \end{aligned}$$

$$p(\mathbf{z}, \mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underbrace{[(\mathbf{z} - \mathbf{H}\mathbf{x})^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{y})^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{y})]}_{=:(*)} \right\}$$

auxilliary quantities:  $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned} (*) &= (\mathbf{A}\mathbf{u})^\top \mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + (\mathbf{B}(\mathbf{u} - \mathbf{y}'))^\top \mathbf{P}^{-1}(\mathbf{B}(\mathbf{u} - \mathbf{y}')) \\ &= \mathbf{u}^\top (\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A}) \mathbf{u} + (\mathbf{u} - \mathbf{y}')^\top (\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}) (\mathbf{u} - \mathbf{y}') \quad \text{Ausmultiplizieren!} \\ &= \mathbf{u}^\top \underbrace{(\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A} + \mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B})}_{=:\mathbf{U}^{-1}} \mathbf{u} - 2\mathbf{u}^\top \underbrace{(\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}) \mathbf{y}'}_{=:\mathbf{U}^{-1}\mathbf{v}} + \text{const.} \\ &= \mathbf{u}^\top \mathbf{U}^{-1} \mathbf{u} - 2\mathbf{u}^\top \mathbf{U}^{-1} \mathbf{v} + \mathbf{v}^\top \mathbf{U}^{-1} \mathbf{v} - \mathbf{v}^\top \mathbf{U}^{-1} \mathbf{v} + \text{const.} \\ &= (\mathbf{u} - \mathbf{v})^\top \mathbf{U}^{-1} (\mathbf{u} - \mathbf{v}) + \text{const.}' \quad \text{(completion of the squares)} \end{aligned}$$

$$p(\mathbf{z}, \mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underbrace{[(\mathbf{z} - \mathbf{H}\mathbf{x})^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{y})^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{y})]}_{=:(*)} \right\}$$

auxiliary quantities:  $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned} (*) &= (\mathbf{A}\mathbf{u})^\top \mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + (\mathbf{B}(\mathbf{u} - \mathbf{y}'))^\top \mathbf{P}^{-1}(\mathbf{B}(\mathbf{u} - \mathbf{y}')) \\ &= \mathbf{u}^\top (\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A}) \mathbf{u} + (\mathbf{u} - \mathbf{y}')^\top (\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}) (\mathbf{u} - \mathbf{y}') \quad \text{multiply!} \\ &= \mathbf{u}^\top \underbrace{(\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A} + \mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B})}_{=:\mathbf{U}^{-1}} \mathbf{u} - 2\mathbf{u}^\top \underbrace{(\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B})}_{=:\mathbf{U}^{-1}\mathbf{v}} \mathbf{y}' + \text{const.} \\ &= \mathbf{u}^\top \mathbf{U}^{-1} \mathbf{u} - 2\mathbf{u}^\top \mathbf{U}^{-1} \mathbf{v} + \mathbf{v}^\top \mathbf{U}^{-1} \mathbf{v} - \mathbf{v}^\top \mathbf{U}^{-1} \mathbf{v} + \text{const.} \\ &= (\mathbf{u} - \mathbf{v})^\top \mathbf{U}^{-1} (\mathbf{u} - \mathbf{v}) + \text{const.}' \quad \text{(completion of the squares)} \end{aligned}$$

$$\begin{aligned} \mathbf{U} &= [\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A} + \mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}]^{-1} = \left[ \begin{pmatrix} \mathbf{R}^{-1} \\ -\mathbf{H}^\top \mathbf{R}^{-1} \end{pmatrix} (\mathbf{I}, -\mathbf{H}) + \begin{pmatrix} \mathbf{O} \\ \mathbf{P}^{-1} \end{pmatrix} (\mathbf{O}, \mathbf{I}) \right]^{-1} \\ &= \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1} \mathbf{H} \\ -\mathbf{H}^\top \mathbf{R}^{-1} & \mathbf{H}^\top \mathbf{R}^{-1} \mathbf{H} + \mathbf{P}^{-1} \end{pmatrix}^{-1} \end{aligned}$$

$$\mathbf{v} = \mathbf{U}(\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B})\mathbf{y}' = \mathbf{U} \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{P}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{O} \\ \mathbf{y} \end{pmatrix} = \mathbf{U} \begin{pmatrix} \mathbf{O} \\ \mathbf{P}^{-1} \mathbf{y} \end{pmatrix}$$



# What did we obtain so far?

$$p(\mathbf{z}, \mathbf{x}) = \mathcal{N}(\mathbf{u}; \mathbf{v}, \mathbf{U}), \quad \text{where: } \mathbf{u} = (\mathbf{z}^\top, \mathbf{x}^\top)^\top$$

$$\text{with: } \mathbf{U} = \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{H} \\ -\mathbf{H}^\top\mathbf{R}^{-1} & \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H} + \mathbf{P}^{-1} \end{pmatrix}^{-1}$$
$$\mathbf{v} = \mathbf{U} \begin{pmatrix} \mathbf{O} \\ \mathbf{P}^{-1}\mathbf{y} \end{pmatrix}$$

- Show that  $p(\mathbf{z}, \mathbf{x})$  is a Gaussian:  $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\left(\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix}; \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{S} & \mathbf{HP} \\ \mathbf{PH}^\top & \mathbf{P} \end{pmatrix}\right)!$
- Calculate from  $p(\mathbf{z}, \mathbf{x})$  the marginal and conditional densities  $p(\mathbf{z})$ ,  $p(\mathbf{x}|\mathbf{z})!$

$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \quad \text{and} \quad p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z}, \mathbf{x})}{p(\mathbf{z})}$$

- Therefore:  $p(\mathbf{z}|\mathbf{x}) p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z}) p(\mathbf{z})!$

Inverse of a partitioned symmetric matrix:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}$$

$$\text{with: } \mathbf{S} = \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C}$$

“*Schur Complement* of matrix  $\mathbf{A}$ ”

proof: verify!

Inverse of a partitioned symmetric matrix:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}$$

$$\text{with: } \mathbf{S} = \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C}$$

“*Schur Complement* of matrix  $\mathbf{A}$ ”

proof: verify!

alternative formulation:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{T}^{-1} & -\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{C}^\top\mathbf{T}^{-1} & \mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{C}^\top\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix}$$

$$\text{mit: } \mathbf{T} = \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top$$

proof: verify!

Inverse of a partitioned symmetric matrix:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}$$

$$\text{mit: } \mathbf{S} = \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C}$$

“Schur Complement of the matrix  $\mathbf{A}$ ”

proof: verify!

alternative formulation:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{T}^{-1} & -\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{C}^\top\mathbf{T}^{-1} & \mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{C}^\top\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix}$$

$$\text{mit: } \mathbf{T} = \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top$$

proof: verify!

In particular, the “matrix inversion lemma” holds:

$$\mathbf{(A - CB^{-1}C^\top)^{-1} = A^{-1} + A^{-1}C(B - C^\top A^{-1}C)^{-1}C^\top A^{-1}}$$

$$p(\mathbf{z}, \mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2} \underbrace{[(\mathbf{z} - \mathbf{H}\mathbf{x})^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{x}) + (\mathbf{x} - \mathbf{y})^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{y})]}_{=:(*)} \right\}$$

auxilliary quantities:  $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned} (*) &= (\mathbf{A}\mathbf{u})^\top \mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + (\mathbf{B}(\mathbf{u} - \mathbf{y}'))^\top \mathbf{P}^{-1}(\mathbf{B}(\mathbf{u} - \mathbf{y}')) \\ &= \mathbf{u}^\top (\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A}) \mathbf{u} + (\mathbf{u} - \mathbf{y}')^\top (\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}) (\mathbf{u} - \mathbf{y}') \quad \text{Ausmultiplizieren!} \\ &= \mathbf{u}^\top \underbrace{(\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A} + \mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B})}_{=:\mathbf{U}^{-1}} \mathbf{u} - 2\mathbf{u}^\top \underbrace{(\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B})}_{=:\mathbf{U}^{-1}\mathbf{v}} \mathbf{y}' + \text{const.} \\ &= \mathbf{u}^\top \mathbf{U}^{-1} \mathbf{u} - 2\mathbf{u}^\top \mathbf{U}^{-1} \mathbf{v} + \mathbf{v}^\top \mathbf{U}^{-1} \mathbf{v} - \mathbf{v}^\top \mathbf{U}^{-1} \mathbf{v} + \text{const.} \\ &= (\mathbf{u} - \mathbf{v})^\top \mathbf{U}^{-1} (\mathbf{u} - \mathbf{v}) + \text{const.}' \quad \text{(completion of the squares)} \end{aligned}$$

$$\begin{aligned} \mathbf{U} &= [\mathbf{A}^\top \mathbf{R}^{-1} \mathbf{A} + \mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B}]^{-1} = \left[ \begin{pmatrix} \mathbf{R}^{-1} \\ -\mathbf{H}^\top \mathbf{R}^{-1} \end{pmatrix} (\mathbf{I}, -\mathbf{H}) + \begin{pmatrix} \mathbf{O} \\ \mathbf{P}^{-1} \end{pmatrix} (\mathbf{O}, \mathbf{I}) \right]^{-1} \\ &= \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1} \mathbf{H} \\ -\mathbf{H}^\top \mathbf{R}^{-1} & \mathbf{H}^\top \mathbf{R}^{-1} \mathbf{H} + \mathbf{P}^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{R} + \mathbf{H} \mathbf{P} \mathbf{H}^\top & \mathbf{H} \mathbf{P} \\ \mathbf{P} \mathbf{H}^\top & \mathbf{P} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{S} & \mathbf{H} \mathbf{P} \\ \mathbf{P} \mathbf{H}^\top & \mathbf{P} \end{pmatrix} \end{aligned}$$

$$\mathbf{v} = \mathbf{U}(\mathbf{B}^\top \mathbf{P}^{-1} \mathbf{B})\mathbf{y}' = \mathbf{U} \begin{pmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{P}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{O} \\ \mathbf{y} \end{pmatrix} = \mathbf{U} \begin{pmatrix} \mathbf{O} \\ \mathbf{P}^{-1} \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{H} \mathbf{y} \\ \mathbf{y} \end{pmatrix}$$

also gilt:

$$p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\left(\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix}; \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}\right)$$

(Konstanten: Normierung!)

$$\begin{aligned}
& \mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P}) \\
&= \underbrace{\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{y}, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x}; \mathbf{y} + \mathbf{W}\boldsymbol{\nu}, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x}; \mathbf{Q}(\mathbf{P}^{-1}\mathbf{y} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}), \mathbf{Q}) \end{cases} \\
&\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1} \\
&\mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.
\end{aligned}$$

- Deute  $\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P})$  als Verbunddichte:  $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x}) p(\mathbf{x})!$  ✓
- Zeige, dass  $p(\mathbf{z}, \mathbf{x})$  eine Gauss-Dichte ist:  $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\left(\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix}; \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}\right)!$  ✓
- Bilde aus der Dichte  $p(\mathbf{z}, \mathbf{x})$  die Marginal- und Konditionaldichten  $p(\mathbf{z})$ ,  $p(\mathbf{x}|\mathbf{z})!$

$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \quad \text{und} \quad p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z}, \mathbf{x})}{p(\mathbf{z})}$$

- Also muss die Beziehung gelten:  $p(\mathbf{z}|\mathbf{x}) p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z}) p(\mathbf{z})!$

$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x} \exp\left\{-\frac{1}{2} \underbrace{\left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]^\top \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}^{-1} \left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]}_{=:(*)}\right\}$$

**Idee: Separiere die Integrationsvariable  $\mathbf{z}$  in einer quadratischen Form!**



$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x} \exp\left\{-\frac{1}{2} \underbrace{\left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]^\top \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}^{-1} \left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]}_{=:(*)}\right\}$$

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$$(*) = \begin{pmatrix} \xi \\ \eta \end{pmatrix}^\top \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{H} \\ -\mathbf{H}^\top\mathbf{R}^{-1} & \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H} + \mathbf{P}^{-1} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{mit:} \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}$$

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$$\begin{aligned} (*) &= \begin{pmatrix} \xi \\ \eta \end{pmatrix}^\top \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{H} \\ -\mathbf{H}^\top\mathbf{R}^{-1} & \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H} + \mathbf{P}^{-1} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{mit: } \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix} \\ &= \xi^\top \mathbf{R}^{-1} \xi - 2\eta^\top \underbrace{\mathbf{H}^\top \mathbf{R}^{-1} \xi}_{=:\mathbf{Q}^{-1}\gamma} + \eta^\top \underbrace{(\mathbf{H}^\top \mathbf{R}^{-1} \mathbf{H} + \mathbf{P}^{-1})}_{=:\mathbf{Q}^{-1}} \eta \end{aligned}$$

$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x} \exp\left\{-\frac{1}{2} \underbrace{\left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]^\top \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}^{-1} \left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]}_{=:(*)}\right\}$$

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$$= \xi^\top \mathbf{R}^{-1} \xi - 2\eta^\top \underbrace{\mathbf{H}^\top \mathbf{R}^{-1} \xi}_{=:\mathbf{Q}^{-1}\gamma} + \eta^\top \underbrace{(\mathbf{H}^\top \mathbf{R}^{-1} \mathbf{H} + \mathbf{P}^{-1})}_{=:\mathbf{Q}^{-1}} \eta$$

$$= \xi^\top \mathbf{R}^{-1} \xi - \gamma^\top \mathbf{Q}^{-1} \gamma + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma) \quad \text{binomische Formel, quadr.Erg.!}$$

$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x} \exp\left\{-\frac{1}{2} \underbrace{\left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]^\top \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}^{-1} \left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]}_{=:(*)}\right\}$$

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$$= \xi^\top \mathbf{R}^{-1} \xi - \xi^\top (\mathbf{R}^{-1} \mathbf{H} \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1}) \xi + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma), \quad \gamma = \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1}$$

$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x} \exp\left\{-\frac{1}{2} \underbrace{\left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]^\top \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}^{-1} \left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]}_{=:(*)}\right\}$$

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$$= \xi^\top \mathbf{R}^{-1} \xi - 2 \underbrace{\eta^\top \mathbf{H}^\top \mathbf{R}^{-1} \xi}_{=:\mathbf{Q}^{-1}\gamma} + \underbrace{\eta^\top (\mathbf{H}^\top \mathbf{R}^{-1} \mathbf{H} + \mathbf{P}^{-1}) \eta}_{=:\mathbf{Q}^{-1}}$$

$$= \xi^\top \mathbf{R}^{-1} \xi - \gamma^\top \mathbf{Q}^{-1} \gamma + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma) \quad \text{binomische Formel, quadr.Erg.!}$$

$$= \xi^\top \mathbf{R}^{-1} \xi - \xi^\top (\mathbf{R}^{-1} \mathbf{H} \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1}) \xi + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma), \quad \gamma = \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1}$$

$$= \xi^\top (\mathbf{R}^{-1} - \mathbf{P}^{-1} \mathbf{H} \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1}) \xi + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma)$$

$$p(\mathbf{z}) = \int d\mathbf{x} p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x} \exp\left\{-\frac{1}{2} \underbrace{\left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]^\top \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}^{-1} \left[\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix} - \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}\right]}_{=:(*)}\right\}$$

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$$= \xi^\top \mathbf{R}^{-1} \xi - \gamma^\top \mathbf{Q}^{-1} \gamma + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma) \quad \text{binomische Formel, quadr.Erg.!}$$

$$= \xi^\top \mathbf{R}^{-1} \xi - \xi^\top (\mathbf{R}^{-1} \mathbf{H} \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1}) \xi + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma), \quad \gamma = \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1}$$

$$= \xi^\top (\mathbf{R}^{-1} - \mathbf{P}^{-1} \mathbf{H} \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1}) \xi + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma)$$

$$= \xi^\top (\mathbf{R} + \mathbf{H} \mathbf{P} \mathbf{H}^\top)^{-1} \xi + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma) \quad \text{Inversionslemma!}$$

$$= \xi^\top \mathbf{S}^{-1} \xi + (\eta - \gamma)^\top \mathbf{Q}^{-1} (\eta - \gamma)$$

$$\begin{aligned}
p(\mathbf{x}) &= \int d\mathbf{y} p(\mathbf{x}, \mathbf{y}) \\
&\propto \exp\left[-\frac{1}{2}(\boldsymbol{\xi}^\top \mathbf{S}^{-1} \boldsymbol{\xi})\right] \int d\mathbf{y} \exp\left[-\frac{1}{2}(\boldsymbol{\eta} - \boldsymbol{\gamma})^\top \mathbf{Q}^{-1}(\boldsymbol{\eta} - \boldsymbol{\gamma})\right] \\
&\propto \exp\left[-\frac{1}{2}(\mathbf{z} - \mathbf{H}\mathbf{y})^\top \mathbf{S}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{y})\right]
\end{aligned}$$

also gilt:

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{y}, \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R})$$

(die Konstanten ergeben sich aus der Forderung:  $\int d\mathbf{x} p(\mathbf{x}) = 1$ )

$$p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z}, \mathbf{x})}{p(\mathbf{z})} \propto \exp\left[-\frac{1}{2}(\boldsymbol{\eta} - \boldsymbol{\gamma})^\top \mathbf{Q}^{-1}(\boldsymbol{\eta} - \boldsymbol{\gamma})\right]$$

$$\begin{aligned} \mathbf{Q} &= \underbrace{(\mathbf{H}^\top \mathbf{R}^{-1} \mathbf{H} + \mathbf{P}^{-1})^{-1}}_{\text{2. Version!}} = \mathbf{P} - \mathbf{P} \mathbf{H}^\top \mathbf{S}^{-1} \mathbf{H} \mathbf{P} \quad \text{Inversionslemma!} \\ &= \underbrace{\mathbf{P} - \mathbf{W} \mathbf{S} \mathbf{H}^\top}_{\text{1. Version!}} = \underbrace{(\mathbf{I} - \mathbf{W} \mathbf{H}) \mathbf{P}}_{\text{3. Version!}}, \quad \text{mit: } \mathbf{W} = \mathbf{P} \mathbf{H}^\top \mathbf{S}^{-1} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\gamma} &= \mathbf{Q} \mathbf{H}^\top \mathbf{R}^{-1} \boldsymbol{\xi} = (\mathbf{I} - \mathbf{W} \mathbf{H}) \mathbf{P} \mathbf{H}^\top \mathbf{R}^{-1} (\mathbf{z} - \mathbf{H} \mathbf{y}) \\ &= (\mathbf{P} \mathbf{H}^\top \mathbf{R}^{-1} - \mathbf{W} \mathbf{H} \mathbf{P} \mathbf{H}^\top \mathbf{R}^{-1}) (\mathbf{z} - \mathbf{H} \mathbf{y}) \\ &= \mathbf{P} \mathbf{H}^\top (\mathbf{R}^{-1} - \mathbf{S}^{-1} (\mathbf{S} - \mathbf{R}) \mathbf{R}^{-1}) (\mathbf{z} - \mathbf{H} \mathbf{y}) = \mathbf{W} (\mathbf{z} - \mathbf{H} \mathbf{y}) \\ &= \mathbf{Q} (\mathbf{P}^{-1} \mathbf{y} + \mathbf{H}^\top \mathbf{R}^{-1} \mathbf{z}) \quad \text{2. Version, Inversionslemma!} \end{aligned}$$

also gilt:

$$p(\mathbf{x}|\mathbf{z}) = \mathcal{N}(\mathbf{x}; \mathbf{q}, \mathbf{Q})$$

$$\mathbf{q} = \mathbf{y} + \mathbf{W} (\mathbf{z} - \mathbf{H} \mathbf{y}), \quad \mathbf{Q} = \mathbf{Y} - \mathbf{W} \mathbf{S} \mathbf{W}^\top$$

(die Konstanten ergeben sich aus der Forderung:  $\int d\mathbf{x} p(\mathbf{x}|\mathbf{z}) = 1$ )