Advanced Sensor Data Fusion in Distributed Systems

SS 2019

Felix Govaers,

Dr. rer. nat., Dipl.-Math.

UNIVERSITÄT BONN

Fraunhofer FKIE

Derivation of the

KALMAN FILTER FORMULAS

Derivation of the Kalman Filter

From the Inversion Lemma, we know that

$$\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}(\mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C} - \mathbf{B})^{-1}\mathbf{C}^\top\mathbf{A}^{-1} = (\mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top)^{-1}$$

An application on the posterior covariance of an estimate yields:

$$\begin{aligned}
\mathbf{P}_{k|k} &= (\mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{H}_k)^{-1} \\
&= \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}\mathbf{H}_k^\top(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_{k|k-1} \\
&= \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}_k^{-1}\mathbf{H}_k\mathbf{P}_{k|k-1} \\
&= \mathbf{P}_{k|k-1} - \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}_k^{-1}\mathbf{S}_k\mathbf{S}_k^{-1}\mathbf{H}_k\mathbf{P}_{k|k-1} \\
\mathbf{W}_{k|k-1} &:= \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}_k^{-1} \\
\mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_k\mathbf{W}_{k|k-1}^\top
\end{aligned}$$

..and for the state

$$-\mathbf{B}^{-1}\mathbf{C}^\top(\mathbf{A}-\mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top)^{-1} = (\mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C}-\mathbf{B})^{-1}\mathbf{C}^\top\mathbf{A}^{-1}$$ (IL)

$$
\begin{aligned}
\mathbf{x}_{k|k} &= \mathbf{P}_{k|k}(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{z}_k) \\
&= \mathbf{P}_{k|k}\left((\mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{H}_k - \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{H}_k)\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{z}_k\right) \\
&= \underbrace{\mathbf{P}_{k|k}\underbrace{(\mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{H}_k)}_{=\mathbf{P}_{k|k}^{-1}}}_{=\mathbf{I}}\mathbf{x}_{k|k-1} - \mathbf{P}_{k|k}\mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{H}_k\mathbf{x}_{k|k-1} + \mathbf{P}_{k|k}\mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{z}_k \\
&= \mathbf{x}_{k|k-1} - \mathbf{P}_{k|k}\mathbf{H}_k^\top\mathbf{R}_k^{-1}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}) \\
&\underset{\mathrm{IL}}{=} \mathbf{P}_{k|k-1}\mathbf{H}_k^\top(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k)^{-1}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}) \\
&= \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}_k^{-1}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}) \\
&= \mathbf{W}_{k|k-1}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1})
\end{aligned}
$$

Kalman Filtering

Putting it all together, we have:

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor model}]{\text{measurement } \mathbf{z}_k} p(\mathbf{x}_k|\mathcal{Z}^{k})$$

$$\mathcal{N}\left(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\right) \xrightarrow[\mathcal{N}\left(\mathbf{z}_k;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k\right)]{} \mathcal{N}\left(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\right)$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_k$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_k\mathbf{W}_{k|k-1}^{\top}$$

$$\boldsymbol{\nu} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$

$$\mathbf{S}_k = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^{\top} + \mathbf{R}_k$$

$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k^{\top}\mathbf{S}_k^{-1}$$

Innovation

Innovation covariance

Kalman gain

Filtering

Input

$\mathbf{x}_{k|k-1}$

$\mathbf{P}_{k|k-1}$

z_k

$\mathbf{R}_k$

Likelihood

Output

$\mathbf{x}_{k|k}$

$\mathbf{P}_{k|k}$

Least Squares

$$\mathbf{P}_{k|k} = \left(\mathbf{P}_{k|k-1}^{-1} - \mathbf{H}_k^\top \mathbf{R}_k^{-1} \mathbf{H}_k\right)^{-1}$$

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}\left(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{z}_k\right)$$

Least Squares
+ Inversion
Lemma

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_k$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_k\mathbf{W}_{k|k-1}^\top$$

$$\boldsymbol{\nu} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$

$$\mathbf{S}_k = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k$$

$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}_k^{-1}$$

Kalman Filter

Simulated Example of a linear Gaussian Sensor and a coordinated turn model.

Intermediate Result

PRODUCT FORMULA

Derivation of the Posterior PDF

The solution is given by BAYES theorem:

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{p(\mathbf{z}_k|\mathbf{x}_k, \mathcal{Z}^{k-1}) \cdot p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int \mathrm{d}\mathbf{x}_k p(\mathbf{z}_k|\mathbf{x}_k, \mathcal{Z}^{k-1}) \cdot p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{p(\mathbf{z}_k|\mathbf{x}_k) \cdot p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int \mathrm{d}\mathbf{x}_k p(\mathbf{z}_k|\mathbf{x}_k) \cdot p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

Prior PDF

Sensor model / Likelihood function

$$= p(\mathbf{z}_k|\mathcal{Z}^{k-1})$$

$$p(\mathbf{x}_k|\mathcal{Z}^k) \cdot p(\mathbf{z}_k|\mathcal{Z}^{k-1}) = p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \cdot p(\mathbf{z}_k|\mathbf{x}_k)$$

Kalman Filter Assumptions

$$p(\mathbf{x}_k|\mathcal{Z}^k) \cdot p(\mathbf{z}_k|\mathcal{Z}^{k-1}) = p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \cdot p(\mathbf{z}_k|\mathbf{x}_k)$$

The **likelihood** is given by a Gaussian linear pdf:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}\left(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\right)$$

We computed the **prior** as

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \mathcal{N}\left(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\right)$$

The **posterior** is given by

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \mathcal{N}\left(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\right)$$

Assume the **expected measurement** also is a Gaussian:

$$p(\mathbf{z}_k|\mathcal{Z}^{k-1}) = \mathcal{N}\left(\mathbf{z}_k;\, \mathbf{z}_{k|k-1},\, \mathbf{S}_k\right)$$

$$\mathcal{N}\left(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\right) \cdot \mathcal{N}\left(\mathbf{z}_k;\, \mathbf{z}_{k|k-1},\, \mathbf{S}_k\right) = \mathcal{N}\left(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\right) \cdot \mathcal{N}\left(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\right)$$

Expected Measurement

The expected measurement is the following Gaussian

$$p(\mathbf{z}_k|\mathcal{Z}^{k-1}) = \mathcal{N}\left(\mathbf{z}_k;\, \mathbf{z}_{k|k-1},\, \mathbf{S}_k\right)$$

with mean

$$\begin{aligned}\mathbf{z}_{k|k-1} &= \mathrm{E}\left[z_k|\mathcal{Z}^{k-1}\right] \\ &= \mathrm{E}\left[\mathbf{H}_k\mathbf{x}_k + v_k|\mathcal{Z}^{k-1}\right] \\ &= \mathbf{H}_k\,\mathrm{E}\left[\mathbf{x}_k|\mathcal{Z}^{k-1}\right] \\ &= \mathbf{H}_k\mathbf{x}_{k|k-1},\end{aligned}$$

and covariance

$$\begin{aligned}\mathbf{S}_{k|k-1} &= \mathrm{cov}\left[\mathbf{z}_k|\mathcal{Z}^{k-1}\right] \\ &= \mathrm{cov}\left[\mathbf{H}_k\mathbf{x}_k + v_k|\mathcal{Z}^{k-1}\right] \\ &= \mathbf{H}_k\,\mathrm{cov}\left[\mathbf{x}_k|\mathcal{Z}^{k-1}\right]\,\mathbf{H}_k^\top + \mathrm{cov}\left[v_k\right] \\ &= \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k.\end{aligned}$$

Product Formula (1st & 2nd Formulation)

We have shown that

$$\mathcal{N}(\mathbf{x};\, \bar{\mathbf{y}},\, \bar{\mathbf{P}}) \cdot \mathcal{N}(\mathbf{z};\, \bar{\mathbf{z}},\, \mathbf{S}) = \mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) \cdot \mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})$$

where

$$\bar{\mathbf{P}} = \begin{cases} \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top \\ (\mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H})^{-1} \end{cases}$$

$$\bar{\mathbf{y}} = \begin{cases} \mathbf{y} + \mathbf{W}\boldsymbol{\nu} \\ \mathbf{P}^{-1}\mathbf{y} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z} \end{cases}$$

$$\mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}$$

$$\mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}$$

Calculating back in time

RETRODICTION

..sometimes also called “Smoothing”

How to update older state estimates?

The history of tracks can be very important. How can we add current information to past states?

$$p(\mathbf{x}_l|\mathcal{Z}^k) \underset{\text{dynamics model}}{\longleftarrow} p(\mathbf{x}_k|\mathcal{Z}^k)$$

We calculate:

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \int \mathrm{d}\mathbf{x}_{l+1}\ \underbrace{p(\mathbf{x}_l|\mathbf{x}_{l+1}, \mathcal{Z}^k)}_{=p(\mathbf{x}_l|\mathbf{x}_{l+1}, \mathcal{Z}^l)}\ p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Retrodiction result of previous “iteration”

Apply Bayes Formula

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \int \mathrm{d}\mathbf{x}_{l+1}\; \underbrace{p(\mathbf{x}_l|\mathbf{x}_{l+1},\mathcal{Z}^k)}_{=p(\mathbf{x}_l|\mathbf{x}_{l+1},\mathcal{Z}^l)}\; p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

$\mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$

$$p(\mathbf{x}_l|\mathbf{x}_l+1,\mathcal{Z}^l) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_l)\; p(\mathbf{x}_l|\mathcal{Z}^l)}{\int \mathrm{d}\mathbf{x}_l\; p(\mathbf{x}_{l+1}|\mathbf{x}_l)\; p(\mathbf{x}_l|\mathcal{Z}^l)}$$

Filtering result at time *l*

Dynamics model *l* -> *l+1*

$\mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l}\mathbf{x}_l,\, \mathbf{Q}_{l+1|l}\big)$

$\mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|l},\, \mathbf{P}_{l|l}\big)$

Fill in the models and compute...

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \int \mathrm{d}\mathbf{x}_{l+1} \frac{\mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l}\mathbf{x}_l,\, \mathbf{Q}_{l+1|l}\big)\; \mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|l},\, \mathbf{P}_{l|l}\big)}{\int \mathrm{d}\mathbf{x}_l\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l}\mathbf{x}_l,\, \mathbf{Q}_{l+1|l}\big)\; \mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|l},\, \mathbf{P}_{l|l}\big)} \cdot \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$$

We have:

$$\frac{\mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l}\mathbf{x}_l,\, \mathbf{Q}_{l+1|l}\big)\; \mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|l},\, \mathbf{P}_{l|l}\big)}{\int \mathrm{d}\mathbf{x}_l\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l}\mathbf{x}_l,\, \mathbf{Q}_{l+1|l}\big)\; \mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|l},\, \mathbf{P}_{l|l}\big)}$$

$$= \frac{\mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big)}{\mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big)} \cdot \mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1} - \mathbf{x}_{l+1|l}),\, \mathbf{P}_{l|l} - \mathbf{W}_{l|l+1}\mathbf{P}_{l+1|l}\mathbf{W}_{l|l+1}^\top\big)$$

where

$$\mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$

"Retrodiction Gain Matrix"

Therefore:

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \int \mathrm{d}\mathbf{x}_{l+1}\, \mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1} - \mathbf{x}_{l+1|l}),\, \mathbf{P}_{l|l} - \mathbf{W}_{l|l+1}\mathbf{P}_{l+1|l}\mathbf{W}_{l|l+1}^\top\big)\; \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$$

Apply Product Formula one more time

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \int \mathrm{d}\mathbf{x}_{l+1}\,\mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1} - \mathbf{x}_{l+1|l}),\, \mathbf{P}_{l|l} - \mathbf{W}_{l|l+1}\mathbf{P}_{l+1|l}\mathbf{W}_{l|l+1}^\top\big)\ \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$$
$$= \mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}\big)$$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l})$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}\left(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l}\right)\mathbf{W}_{l|l+1}^\top$$
$$\mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$

"Rauch-Tung-Striebel"-Equations

where

Rauch-Tung-Striebel Retrodiction

The track of a target is smoothed by the RTS equations. The trajectory becomes much nicer and the estimation quality is improved.