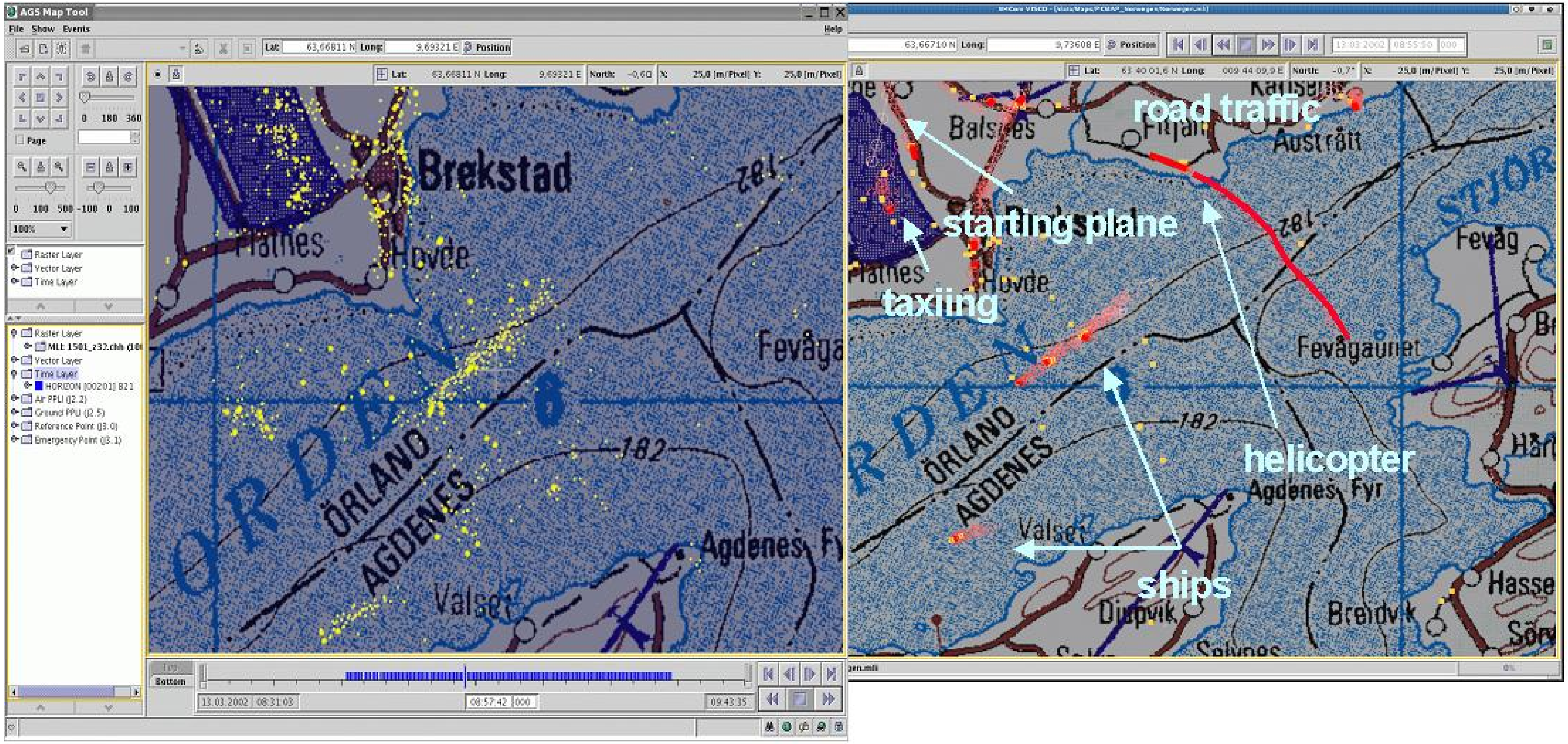
Unfortunately, the Lecture on December 18 will NOT take place!

integration of available context information

- **GMTI radar: Doppler blindness (MDV)**
- **road-map and terrain information (GIS)**

Examples of GMTI Tracks (live exercise)

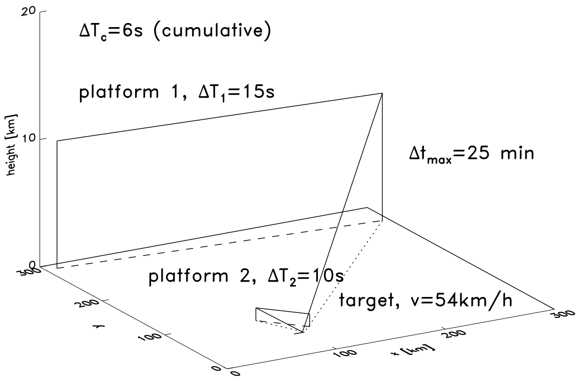


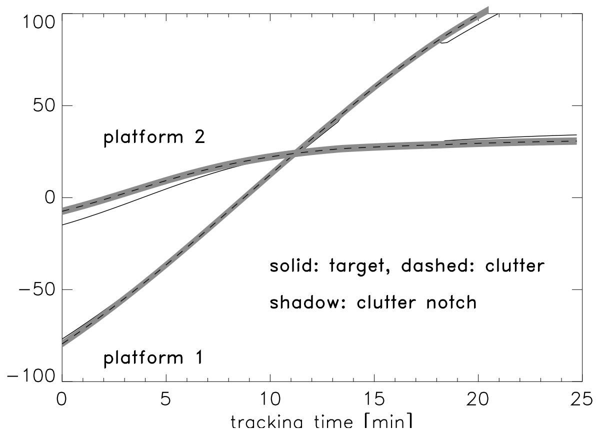
Fraunhofer FKIE

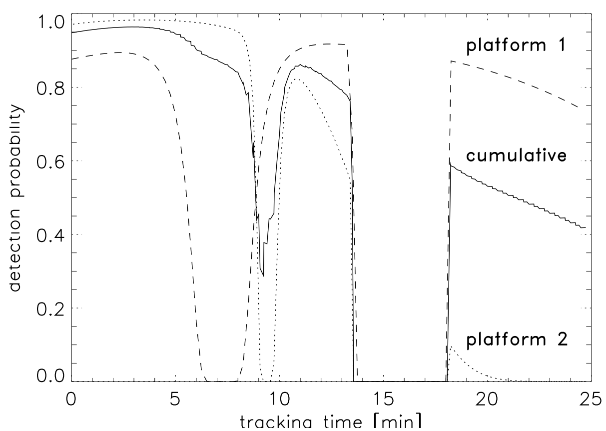
Cumulative Detection by N Sensors

cumulative **detection probability** $$P_D^{\text{kum}}(N) = 1 - \prod_{n=1}^{N} (1 - P_D^n)$$

example: Doppler blindness in case of GMTI radar







mean cumulative revisit interval: $$1/\Delta T_c = \sum_{n=1}^{N} 1/\Delta T_n$$

mean cumulative P_D relative to ΔT_c: $$P_D^c = 1 - \prod_{n=1}^{N} (1 - P_D^n)^{\Delta T_c/\Delta T_n}$$

GMTI Radar: Ground Moving Target Indicator Radar

low-DOPPLER targets can be masked by the GMTI clutter notch

- *fading:* series of missing plots (target/sensor geometry)
- *stopping targets:* indistinguishable from ground clutter
- *mdv:* minimum detectable velocity (sensor parameter)

a simple GMTI detection model: qualitative discussion

- detection depends on target kinematics & target/sensor geometry
- detection probability $P_D(\mathrm{x}_k)$ is small if $\dot{r}_k - \dot{r}_{\text{mlc}}(\mathrm{x}_k) < \textit{mdv}$
- there exists a narrow transition region between these domains
- the state-dependent $P_D(\mathrm{x}_k)$ is part of the likelihood function

Fraunhofer FKIE

Recapitulation: The Detection Process

Detector: receives signals and decides on object existence

Processor: processes detected signals and produces measurements

‘D’: detector detects an object

D: object actually existent

error of 1. kind: $P_{\mathrm{I}} = P(\neg\text{‘}D\text{’}|D)$

error of 2. kind: $P_{\mathrm{II}} = P(\text{‘}D\text{’}|\neg D)$

measure of detection performance: $P_D = P(\text{‘}D\text{’}|D)$

detector properties characterized by two parameters:

- detection probability $P_D = 1 - P_{\mathrm{I}}$
- false alarm probability $P_F = P_{\mathrm{II}}$

Fraunhofer FKIE

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example (Swerling I model): $P_D = P_D(P_F, \text{SNR}) = P_F^{1/(1+\text{SNR})}$

detector design: Maximize detection probability P_D for a given, predefined false alarm probability P_F!

sensor performance: *quantitative* model

- **basis:** $\text{snir} = \text{snir}(r_k, \varphi_k, \dot{r}_k)$, **Signal-to-Noise+Interference Ratio**

$$\text{snir} = \text{snir}_0 \underbrace{\left(\frac{\bar{\sigma}_k}{\sigma_0}\right)}_{\text{rcs}} \underbrace{\left(\frac{r_k}{r_0}\right)^{-4}}_{\text{propagation}} \underbrace{D(\varphi_k)}_{\text{directivity}} \underbrace{\left[1 - \mathrm{e}^{-\log 2 \left(\frac{n_c(r_k,\varphi_k,\dot{r}_k)}{v_m}\right)^2}\right]}_{\text{clutter notch } (<\frac{1}{2} \text{ for } |n_c|<v_m)}$$

- **quadrature detector with given P_{FA}, rcs fluctuations: Swerling I**

$$P_D(r_k, \varphi_k, \dot{r}_k) = P_{\text{FA}}^{\frac{1}{1+\text{snir}}} \approx P_d\left(1 - \mathrm{e}^{-\log 2 \left(\frac{n_c(r_k,\varphi_k,\dot{r}_k)}{v_m}\right)^2}\right)$$

- **as usual: residual clutter; bias free, Gaussian errors (monopulse)**

$$\sigma_{r,\varphi,\dot{r}}(r_k, \varphi_k, \dot{r}_k) = \Sigma_{r,\varphi,\dot{r}} / \sqrt{\text{snir}(r_k, \varphi_k, \dot{r}_k)}$$

recapitulation: ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

$m_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{m_k}$!

- E_0: the object was not detected; m_k false plots in the Field of View (FoV)
- E_j, $j = 1, \ldots, m_k$: object detected; $\mathbf{z}_k^j$: object measurement; $m_k - 1$ false plots

Consider the interpretations in the likelihood function $p(Z_k, m_k|\mathbf{x}_k)$!

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Fraunhofer FKIE

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sensor parameter: detection probability $P_D(\mathbf{x}_k)$

$$P_D(r_k, \varphi_k, \dot{r}_k) \approx P_d\Big(1 - \mathrm{e}^{-\log 2 \left(\frac{n_c(\mathbf{x}_k)}{v_m}\right)^2}\Big)$$

Low-Doppler targets: masked by the GMTI clutter notch.

Unit vector pointing from the platform position $\mathbf{p}_k$ at time t_k
to the target at the position $\mathbf{r}_k$ moving with the velocity $\dot{\mathbf{r}}_k$:

$$\mathbf{e}_k^p = \frac{\mathbf{r}_k - \mathbf{p}_k}{|\mathbf{r}_k - \mathbf{p}_k|}$$

Kinematic state: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$. Doppler blindness occurs if
radial velocities of the object and the “main-lobe clutter” are identical, i.e.

$$h_n(\mathbf{r}_k, \dot{\mathbf{r}}_k; \mathbf{p}_k) = \frac{(\mathbf{r}_k - \mathbf{p}_k)^\top \dot{\mathbf{r}}_k}{|\mathbf{r}_k - \mathbf{p}_k|} \approx 0.$$

Target velocity is nearly orthogonal to sensor-to-target line-of-sight.

recapitulation: ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

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sensor parameter: detection probability $P_D(\mathbf{x}_k)$

$$P_D(r_k, \varphi_k, \dot{r}_k) \approx P_d\Big(1 - 2\pi \mathsf{mdv}\, \mathcal{N}\big(0;\, h_n(\mathbf{x}_k),\, \mathsf{mdv}^2\big)\Big)$$

Effect: Process an additional ficticious measurement,
Gaussian mixtures with possibly negative coefficients.

recapitulation: ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

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$$= \underbrace{p(Z_k|m_k, \neg D, \mathbf{x}_k)}_{=|\text{FoV}|^{-m_k}} \underbrace{p(m_k|\neg D, \mathbf{x}_k)}_{=p_F(m_k)} (1 - P_D) + P_D \sum_{j=1}^{m_k} p(Z_k, m_k, j|D, \mathbf{x}_k)$$

false measurements: Poisson distributed in #, uniformly distributed in the FoV

recapitulation: ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

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$$= p(Z_k|m_k, \neg D, \mathbf{x}_k)\, p(m_k|\neg D, \mathbf{x}_k)\, (1 - P_D) + P_D \sum_{j=1}^{m_k} p(Z_k, m_k, j|D, \mathbf{x}_k)$$

$$= |\text{FoV}|^{-m_k}\, p_F(m_k)\, (1 - P_D) + P_D \sum_{j=1}^{m_k} \underbrace{p(Z_k|m_k, j, D, \mathbf{x}_k)}_{|\text{FoV}|^{-(m_k-1)} \mathcal{N}(\mathbf{z}_k^j; \mathbf{H}\mathbf{x}_k, \mathbf{R})}\; \underbrace{p(j|m_k, D)}_{=1/m_k}\; \underbrace{p(m_k|D)}_{=p_F(m_k-1)}$$

Insert Poisson distribution: $p_F(m_k) = \frac{(\rho_F|\text{FoV}|)^{-m_k}}{m_k!}\, \mathrm{e}^{-\rho_F|\text{FoV}|}$

recapitulation: ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

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$$= |\text{FoV}|^{-m_k}\, p_F(m_k)\, (1 - P_D) + P_D \sum_{j=1}^{m_k} \underbrace{p(Z_k|m_k, j, D, \mathbf{x}_k)}_{|\text{FoV}|^{-(m_k-1)} \mathcal{N}(\mathbf{z}_k^j; \mathbf{H}\mathbf{x}_k, \mathbf{R})}\, \underbrace{p(j|m_k, D)}_{=1/m_k}\, \underbrace{p(m_k|D)}_{=p_F(m_k-1)}$$

$$= \frac{e^{-\rho_F |\text{FoV}|}}{n_k!} \Big(\big((1 - P_D(\mathbf{x}_k)\big) \rho_F^{m_k} + P_D(\mathbf{x}_k)\, \rho_F^{m_k-1} \sum_{j=1}^{m_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big) \Big)$$

Recapitulation: Likelihood Functions

The likelihood function answers the question:

What does the sensor tell about the state x of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1$**,** $\rho_F = 0$

  at each time one measurement:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k; \mathbf{H}\mathbf{x}_k, \mathbf{R})$$

- **real conditions, one object:** $P_D < 1$**,** $\rho_F > 0$

  at each time m_k measurements $Z_k = \{\mathbf{z}_k^1, \dots, \mathbf{z}_k^{n_k}\}$!

$$p(Z_k, m_k|\mathbf{x}_k) \propto \big(1 - P_D(\mathbf{x}_k)\big)\rho_F + P_D(\mathbf{x}_k) \sum_{j=1}^{m_k} \mathcal{N}\big(\mathbf{z}_k^j;\ \mathbf{H}\mathbf{x}_k,\ \mathbf{R}\big)$$

$$\textbf{filtering:} \quad p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor modell}]{\text{sensor data } Z_k} p(\mathbf{x}_k|\mathcal{Z}^k)$$

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{p(Z_k, n_k|\mathbf{x}_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k \underbrace{p(Z_k, n_k|\mathbf{x}_k)}_{\text{likelihood}} \underbrace{p(\mathbf{x}_k|\mathcal{Z}^{k-1})}_{\text{prediction}}} \qquad \textbf{(Bayes)}$$

likelihood (depending on sensor data, modeling parameters):

$$p(Z_k, m_k|\mathbf{x}_k) \propto \pi_0^0 + \sum_{i=0}^{m_k} \sum_{j=0}^{1} \pi_n^i \, \mathcal{N}\Big(\mathbf{z}_k^{ij};\, \mathbf{H}_{k|k-1}^{ij}\mathbf{x}_k,\, \mathbf{R}_{k|k-1}^{ij}\Big)$$

(after some calculations and mild approximations; π_n^i constant coefficients)

essential: exploit state dependency of $P_D(\mathbf{x}_k)$!

Clutter Notch: A Priori Information!

- *current* position (Sensor-to-target-geometry)
- sensor specific *width* (STAP $\rightarrow$ MDV)
- detection process: *generic* model

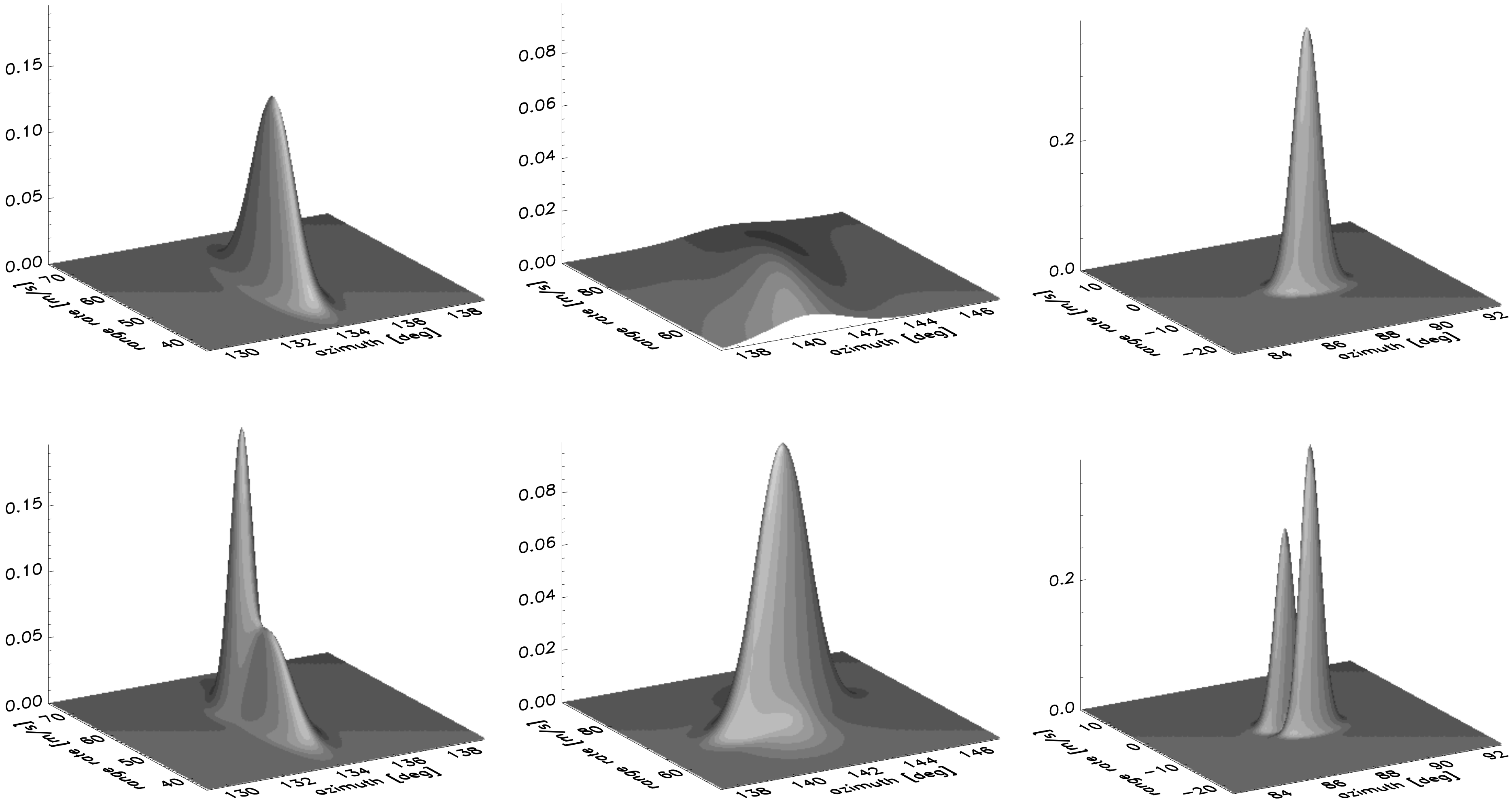
GMTI model $\Rightarrow$ mixture densities:

- well-understood formalism directly applicable
- class of GAUSS*ian mixtures* remains invariant
- pdfs characterized by a *set of parameters*
- growing memory: standard-type approximations

model inherent: **reason for missing detections**

$\Rightarrow$ **An adequate treatment becomes possible!**

PDFs with / without exploiting the GMTI sensor model

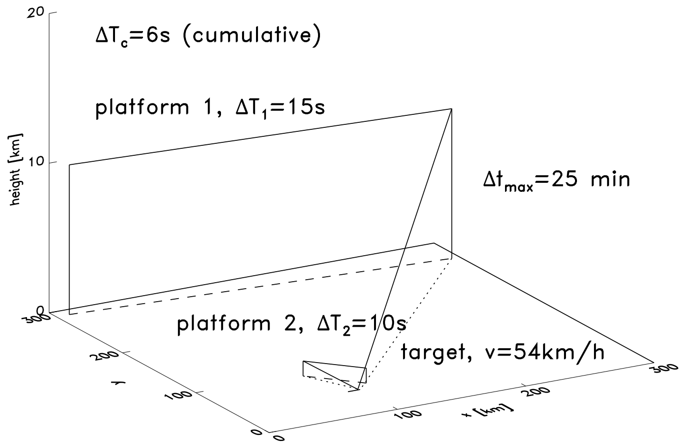


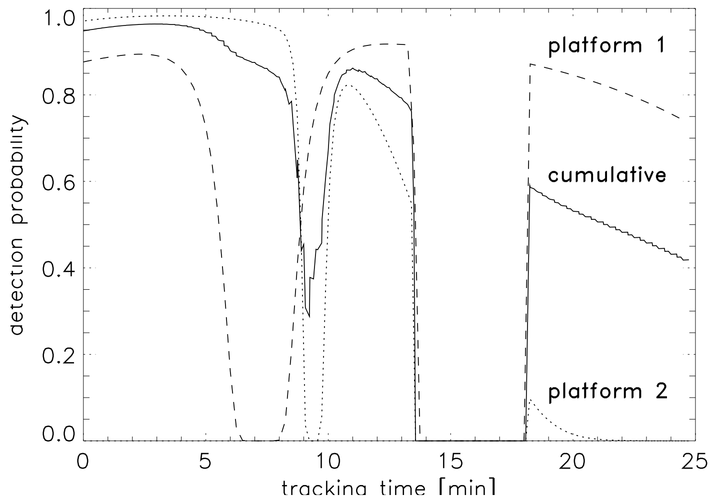
Missing detection occurred near the clutter notch

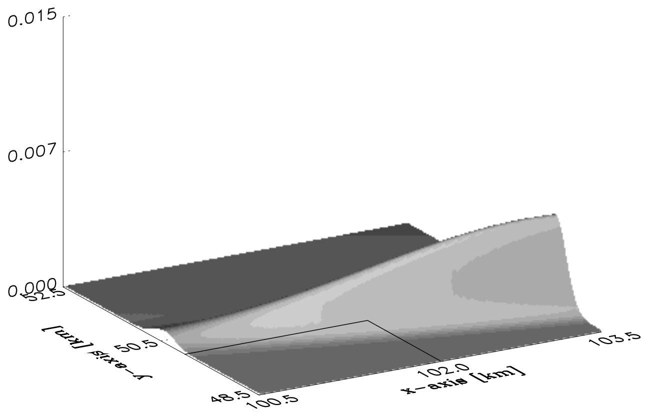
Several missing detections in the clutter notch

Detection occurred near the clutter notch

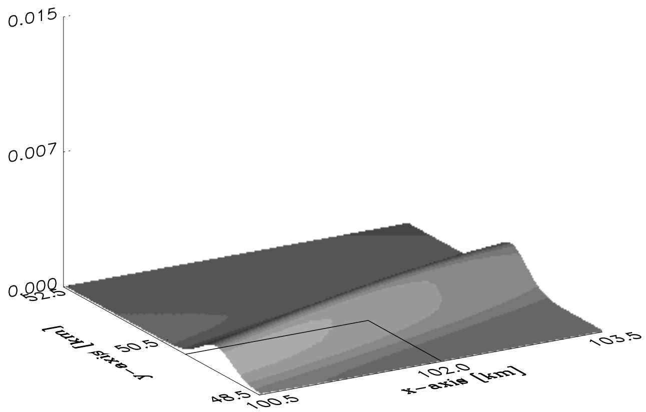
early detection of stopping targets



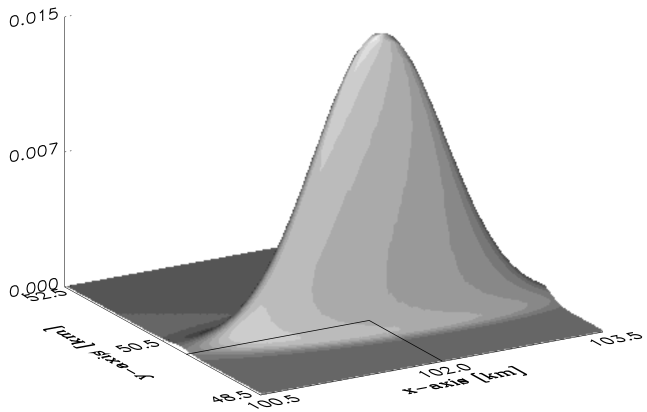




neg. output sensor 1



neg. output sensor 2



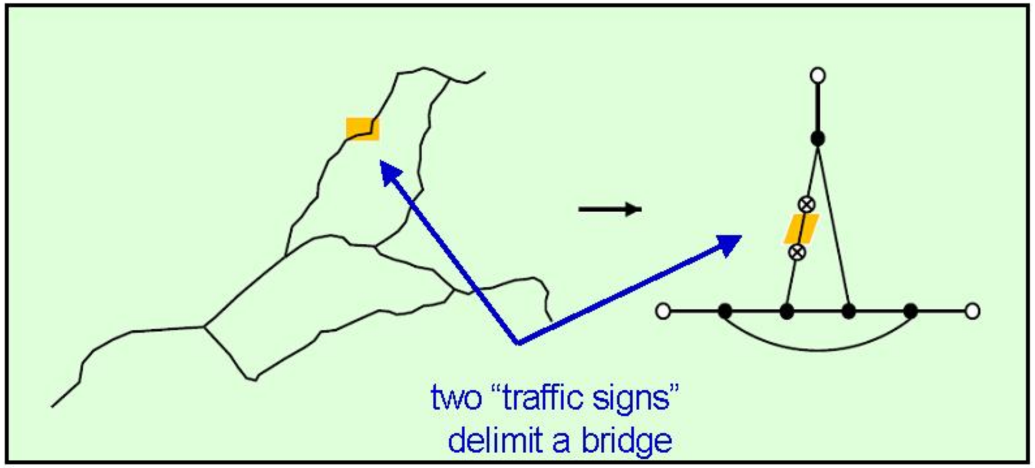
fusion: sensor 1+2

A *'negative' sensor output* can also provide information on the kinematical state vector of a target.

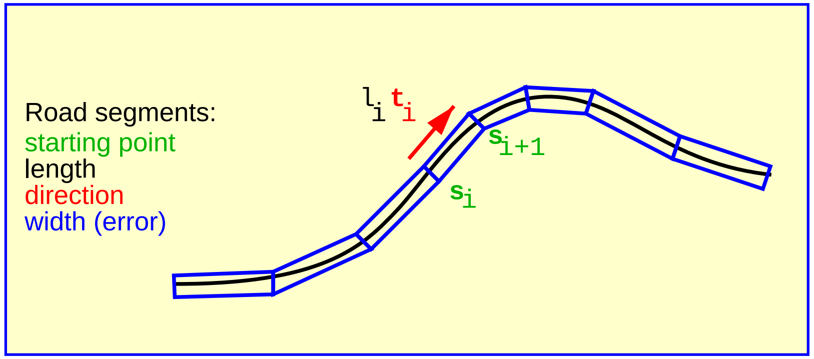
- ***fictitious plot*: function of position / radial speed**
- ***mdv*: appears as a fictitious measurement error**
- ***fusion*: exploit differing target/sensor geometries**

$$\ell(\mathbf{x}_k; Z_k) = \Big(1 - P_D(\mathbf{x}_k)\Big)\rho_F + P_D(\mathbf{x}_k) \sum_{j=1}^{m_k} \mathcal{N}\Big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}_k^j\Big)$$

a simple model for road networks



nodes: sources, crossings, ‘traffic signs’
edges: representation of road segments



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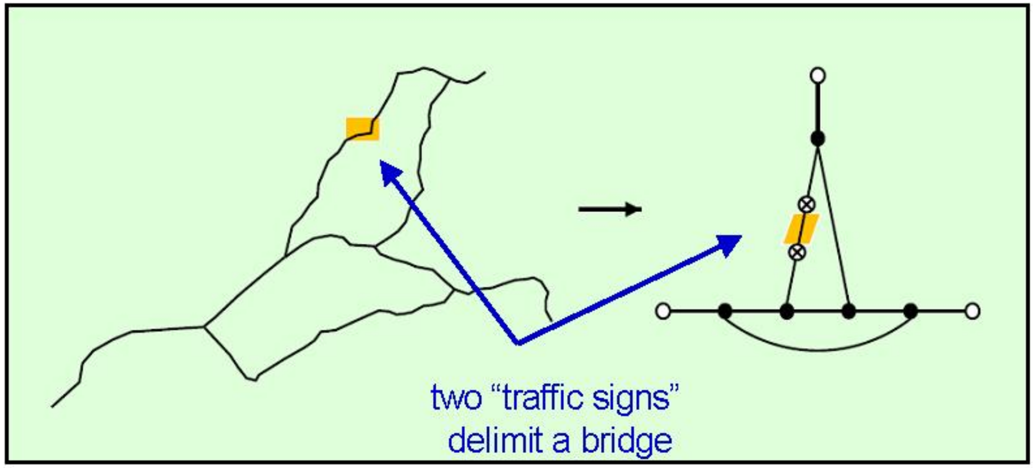
a simple model for road networks



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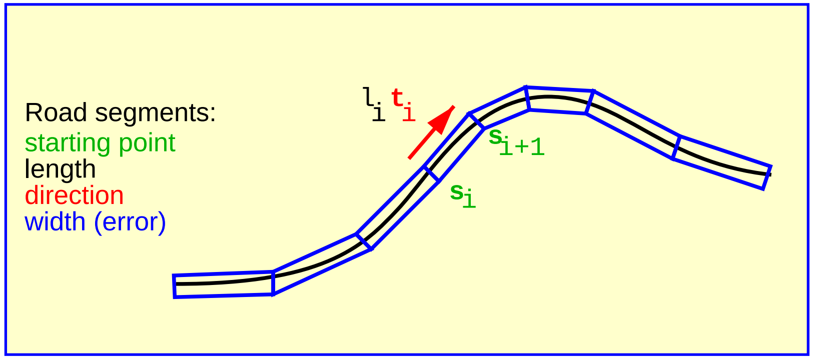

approximation of a road segment: $\mathcal{R} : l \in [l_1, l_{n_r}) \mapsto \mathcal{R}(l) = \sum_{m=1}^{n_r} \left[\mathbf{s}_m + (l - l_m)\mathbf{t}_m \right] \chi_m(l)$

arc length l, node vector $\mathbf{s}_m = \mathcal{R}(l_m)$, tangential vector $\mathbf{t}_m$, # of nodes n_r

accuracy of $\mathbf{s}_m$: covariance matrix $\mathbf{R}_m$, $\chi_m(l) = \begin{cases} 1 & \text{for } l \in [l_m, l_{m+1}) \\ 0 & \text{else} \end{cases}$

in general: $||\mathbf{s}_m - \mathbf{s}_{m-1}|| \leq l_m - l_{m-1} =: \lambda_m$ (measure of discretization errors)

On road-map assisted vehicle tracking

road maps: polygons $\mathcal{R}(l)$ (arc length l, with mapping/discretization errors)

road coordinates: convoy kinematics described by: $\boxed{\mathbf{x}_k^r = (l_k, \dot{l}_k)^\top}$ (2D!)

dynamics: prediction in road coordinates: $p(\mathbf{x}_{k-1}^r|\mathcal{Z}^{k-1}) \xrightarrow{\text{Dyn.}} p(\mathbf{x}_k^r|\mathcal{Z}^{k-1})$

sensor: filtering step in sensor coordinates: $p(\mathbf{x}_k^s|\mathcal{Z}^{k-1}) \xrightarrow[Z_k]{\text{Sen.}} p(\mathbf{x}_k^s|\mathcal{Z}^k)$

$$\boxed{\underbrace{p(\mathbf{x}_k^r|\mathcal{Z}^{k-1})}_{\text{road coordinates}} \quad \xrightarrow[\text{mapping errors } \mathbf{R}_i^j]{\text{road network}} \quad \underbrace{p(\mathbf{x}_k^s|\mathcal{Z}^{k-1})}_{\text{sensor coordinates}}}$$

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- **result: GAUSSIAN mixtures referring to road segments**
- **road segments: essentially artificial scalar measurements**
- **in principle seamlessly embedded into BAYES*ian formalism***
- **road *networks* imply an inherent multihypothesis structure**

Road: continuous 3D curve $\mathcal{R}^*$ in Cartesian coordinates, which is parameterized by its arc length l.

Consider a **piecewise linear approximation** of $\mathcal{R}^* : l \mapsto \mathcal{R}^*(l)$ by a polygonal curve $\mathcal{R}$, characterized by n_r node vectors

$$\mathbf{s}_m = \mathcal{R}^*(l_m), \quad m = 1, \ldots, n_r.$$

Normalized tangential vectors: $\mathbf{t}_m = \dfrac{\mathbf{s}_{m+1} - \mathbf{s}_m}{||\mathbf{s}_{m+1} - \mathbf{s}_m||}, \quad m = 1, \ldots, n_r - 1$

EUCLIDian distance $||\mathbf{s}_{m+1} - \mathbf{s}_m||$ between two nodes usually not identical with the distance $\lambda_m = l_{m+1} - l_m$ actually covered by a vehicle moving from $\mathbf{s}_m$ to $\mathbf{s}_{m+1}$.

Besides $\mathbf{s}_m$, $\lambda_m \geq |\mathbf{s}_{m+1} - \mathbf{s}_m|$ should enter into the road model.
$\sigma_d = |\lambda_m - ||\mathbf{s}_{m+1} - \mathbf{s}_m||\,|$: a measure of the discretization errors.

Polygon curve $\mathcal{R}$, by which the road $\mathcal{R}^*$ is approximated:

$$\mathcal{R} : l \mapsto \mathcal{R}(l) = \sum_{m=0}^{n_r} \Big[\mathbf{s}_m + (l - l_m)\mathbf{t}_m\Big]\, \chi_m(l)$$

with: $\mathcal{R}^*(l_m) = \mathcal{R}(l_m) = \mathbf{s}_m, \quad m = 0, \ldots, n_r.$

With characteristic functions:

$$\chi_m(l) = \begin{cases} 1 & \text{for } l \in (l_m, l_{m+1}) \\ 0 & \text{else} \end{cases}$$

$$m = 0, \ldots, n_r, \quad l_0 = -\infty, \quad l_{n_r+1} = \infty$$

and $\quad \mathbf{s}_0 = \mathbf{s}_1, \quad \mathbf{t}_0 = \mathbf{t}_1, \quad l_0 = l_1, \quad \mathbf{t}_{n_r} = \mathbf{t}_{n_r-1},$

Road-constrained Densities

Find an operator $\mathcal{T}_{g\leftarrow r}$ transforming $p(\mathbf{x}_k^r|\mathcal{Z}^{k-1})$ into ground coordinates:

$$p(\mathbf{x}_k^g|\mathcal{Z}^{k-1}) = \mathcal{T}_{g\leftarrow r}[p(\mathbf{x}_k^r|\mathcal{Z}^{k-1})].$$

Marginalization: write $p(\mathbf{x}_k^g|\mathcal{Z}^{k-1})$ as a sum over the road segments:

$$\begin{aligned} p(\mathbf{x}_k^g|\mathcal{Z}^{k-1}) &= \sum_{m=0}^{n_r} p(\mathbf{x}_k^g, m|\mathcal{Z}^{k-1}) \\ &= \sum_{m=0}^{n_r} p(\mathbf{x}_k^g|m, \mathcal{Z}^{k-1})\, p(m|\mathcal{Z}^{k-1}) \\ &= \sum_{m=0}^{n_r} p_{g\leftarrow r}^m\, \mathcal{T}_{g\leftarrow r}^m\Big[p(\mathbf{x}_k^r|\mathcal{Z}^{k-1})\Big]. \end{aligned}$$

$$\begin{aligned} p(m|\mathcal{Z}^{k-1}) &= \int d\mathbf{x}_k^r\, p(m, \mathbf{x}_k^r|\mathcal{Z}^{k-1}) \\ &= \int d\mathbf{x}_k^r\, \chi_m(\mathbf{H}_r\mathbf{x}_k^r)\, p(\mathbf{x}_k^r|\mathcal{Z}^{k-1}) =: p_{g\leftarrow r}^m \end{aligned}$$

$p_{g\leftarrow r}^m$: probability that the vehicles moves on the segment m given $\mathcal{Z}^{k-1}$ with: $\mathbf{H}_r$ definded by $\mathbf{H}_r\mathbf{x}_k^r = l_k$. Later: a fictitious measurement matrix.

$$p^m_{g\leftarrow r} = \int d\mathbf{x}^r_k \, \chi_m(\mathbf{H}_r \mathbf{x}^r_k) \, p(\mathbf{x}^r_k | \mathcal{Z}^{k-1})$$

Since $p(\mathbf{x}^r_k | \mathcal{Z}^{k-1})$ is a Gaussian or a Gaussian mixture due to the GMTI model, the weighting factors $p^m_{g\leftarrow r}$ can be expressed by error functions:

$$p^m_{g\leftarrow r} = p^j_{k-1} \left(\Phi[\lambda(l^j_{m+1})] - \Phi[\lambda(l^j_m)] \right), \quad m = 0, \ldots, n_r$$

$$\Phi(\lambda) = 1/\sqrt{2\pi} \int_{-\infty}^{\lambda} dt \, \exp(-t^2/2), \quad \lambda(l)^j = \frac{l - \mathbf{H}_r \mathbf{x}^{rj}_{k|k-1}}{\sqrt{\mathbf{H}_r \mathbf{P}^{rj}_{k|k-1} \mathbf{H}_r^\top}}.$$

For the remaining term standard probability reasoning yields:

$$\begin{aligned}
\mathcal{T}^m_{g\leftarrow r}\left[p(\mathbf{x}^r_k | \mathcal{Z}^{k-1}) \right] &= p(\mathbf{x}^g_k | m, \mathcal{Z}^{k-1}) \\
&= \int d\mathbf{x}^r_k \, p(\mathbf{x}^g_k, \mathbf{x}^r_k | m, \mathcal{Z}^{k-1}) \\
&= \int d\mathbf{x}^r_k \, p(\mathbf{x}^g_k | \mathbf{x}^r_k, m) \, p(\mathbf{x}^r_k | m, \mathcal{Z}^{k-1}).
\end{aligned}$$

Limiting case of a straight road: $\mathcal{R}(l) = \mathrm{s} + l\mathrm{t}$.

Consider the transform from road to ground coordinates defined by:

$$p(\mathbf{x}_{k+1}^g | \mathbf{x}_{k+1}^r) = \mathcal{N}\Big(\mathbf{x}_{k+1}^g;\ \mathbf{t}_{g\leftarrow r}[\mathbf{x}_{k+1}^r],\ \sigma_m^2\Big)$$

with the affine transform $\mathbf{t}_{g\leftarrow r}[\mathbf{x}_r] = \left(\begin{smallmatrix} \mathrm{t} & 0 \\ 0 & \mathrm{t} \end{smallmatrix}\right)\mathbf{x}_r + \left(\begin{smallmatrix} \mathrm{s}-l\mathrm{t} \\ 0 \end{smallmatrix}\right)$

σ_m: modelling the mapping error.

Transformation of $p(\mathbf{x}_k^r | \mathcal{Z}^{k-1})$ into the ground coordinate system:

$$p(\mathbf{x}_k^g | \mathcal{Z}^{k-1}) = \int d\mathbf{x}_k^r\, p(\mathbf{x}_k^g | \mathbf{x}_k^r)\, p(\mathbf{x}_k^r | \mathcal{Z}^{k-1}).$$

The integration via product formula, preserving normal mixtures. The corresponding inverse is simply provided by a projection of the density $p(\mathbf{x}_k^g | \mathcal{Z}^{k-1})$ on the road.

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Polygonal Roads

The transition density $p(\mathbf{x}_k^g|\mathbf{x}_k^r, m)$ for the road segment m is characterized by road map and discretization errors (σ_m, σ_d), which may vary from segment to segment.

With affine transformations $\mathbf{t}_{g\leftarrow r}^m[\mathbf{x}_r] = \left(\begin{smallmatrix} \mathbf{t}_m & 0 \\ 0 & \mathbf{t}_m \end{smallmatrix}\right)\mathbf{x}_r + \left(\begin{smallmatrix} \mathbf{s}_m - l_m \mathbf{t}_m \\ 0 \end{smallmatrix}\right)$

for each individual road segment m, and $\sigma_r^2 = \sigma_m^2 + \sigma_d^2$ we consider:

$$p(\mathbf{x}_g|\mathbf{x}_r, m) = \mathcal{N}\Big(\mathbf{x}_g;\ \mathbf{t}_{g\leftarrow r}^m[\mathbf{x}_r],\ \sigma_r^2\Big).$$

Apply Bayes' rule: $$p(\mathbf{x}_k^r|m, \mathcal{Z}^{k-1}) = \frac{p(m|\mathbf{x}_k^r)\, p(\mathbf{x}_k^r|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k^r\, p(m|\mathbf{x}_k^r)\, p(\mathbf{x}_k^r|\mathcal{Z}^{k-1})}$$

with: $$p(m|\mathbf{x}_k^r) = \chi_m(\mathbf{H}_r\mathbf{x}_k^r) \approx \exp[-\tfrac{1}{2}(z_r^m - \mathbf{H}_r\mathbf{x}_r)^2/\lambda_m^2]$$
$$c = \sqrt{2\pi}\lambda_m\ \mathcal{N}\Big(z_r^m;\ \mathbf{H}_r\mathbf{x}_r,\ \lambda_m^2\Big)$$

z_r^m and λ_m^2: mean and variance of a uniform density given by $\chi_m(l)$. Product formula:

$$p(\mathbf{x}_k^r|m, \mathcal{Z}^{k-1}) = p_{k-1}^m\ \mathcal{N}\Big(\mathbf{x}_k^r;\ \mathbf{x}_{k|k-1}^{rmj},\ \mathbf{P}_{k|k-1}^{rmj}\Big)$$

with Kalman-type update equations for $\mathbf{x}_{k|k-1}^{rmj}$ and $\mathbf{P}_{k|k-1}^{rmj}$.

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