$$\mathcal{N}\big(\mathbf{z};\ \mathbf{H}\mathbf{x},\ \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{P}\big)$$

$$= \underbrace{\mathcal{N}\big(\mathbf{z};\ \mathbf{H}\mathbf{y},\ \mathbf{S}\big)}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}\big(\mathbf{x};\ \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\ \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top\big) \\ \mathcal{N}\big(\mathbf{x};\ \mathbf{Q}(\mathbf{P}^{-1}\mathbf{y} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\ \mathbf{Q}\big) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}$$

$$\mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

- Interpret $\mathcal{N}\big(\mathbf{z};\ \mathbf{H}\mathbf{x},\ \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{P}\big)$ as a joint density: $p(\mathbf{z},\mathbf{x}) = p(\mathbf{z}|\mathbf{x})\,p(\mathbf{x})$!

  - Show that $p(\mathbf{z},\mathbf{x})$ is a Gaussian: $p(\mathbf{z},\mathbf{x}) = \mathcal{N}\Big(\big(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\big);\ \big(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\big),\ \big(\begin{smallmatrix}\mathbf{S} & \mathbf{H}\mathbf{P}\\ \mathbf{P}\mathbf{H}^\top & \mathbf{P}\end{smallmatrix}\big)\Big)$!

- Calculate from $p(\mathbf{z},\mathbf{x})$ the marginal and conditional densities $p(\mathbf{z})$, $p(\mathbf{x}|\mathbf{z})$!

$$p(\mathbf{z}) = \int d\mathbf{x}\ p(\mathbf{z},\mathbf{x}) \qquad \text{and} \qquad p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z},\mathbf{x})}{p(\mathbf{z})}$$

- Therefore: $p(\mathbf{z}|\mathbf{x})\,p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})\,p(\mathbf{z})$!

$$p(\mathbf{z},\mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \text{ exp}\Big\{ -\tfrac{1}{2}\underbrace{[(\mathbf{z}-\mathbf{H}\mathbf{x})^\top\mathbf{R}^{-1}(\mathbf{z}-\mathbf{H}\mathbf{x}) + (\mathbf{x}-\mathbf{y})^\top\mathbf{P}^{-1}(\mathbf{x}-\mathbf{y})]}_{=:(*)}\Big\}$$

Strategy: Try to rewrite (*) in such a way, that the state vector $\mathbf{u} = (\mathbf{z}, \mathbf{x})^\top$ appears in a quadratic form $(\mathbf{u}-\mathbf{v})^\top\mathbf{U}^{-1}(\mathbf{u}-\mathbf{v})$.

Do $\mathbf{u}$, $\mathbf{z}$ or $\mathbf{x}$ not appear (*) elsewhere, we have shown that $\mathbf{u}$ is Gaussian distributed with expectation value $\mathbf{v}$ and covariance matrix $\mathbf{U}$. For all multiplicative constants result from knowing that $p(\mathbf{u}) = \mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\ \mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P})$ is a pdf, i.e. correctly normalized.

$$p(\mathrm{z},\mathrm{x}) = (|2\pi\mathrm{R}||2\pi\mathrm{P}|)^{\frac{1}{2}} \ \text{exp}\Big\{ -\tfrac{1}{2}\underbrace{[(\mathrm{z}-\mathrm{Hx})^\top\mathrm{R}^{-1}(\mathrm{z}-\mathrm{Hx}) + (\mathrm{x}-\mathrm{y})^\top\mathrm{P}^{-1}(\mathrm{x}-\mathrm{y})]}_{=:(*)}\Big\}$$

Strategy: Try to rewrite (*) in such a way, that the state vector $\mathrm{u} = (\mathrm{z},\mathrm{x})^\top$ appears in a quadratic form $(\mathrm{u}-\mathrm{v})^\top\mathrm{U}^{-1}(\mathrm{u}-\mathrm{v})$.

Do u, z or x not appear (*) elsewhere, we have shown that u is Gaussian distributed with expectation value v and covariance matrix U. For all multiplicative constants result from knowing that $p(\mathrm{u}) = \mathcal{N}\big(\mathrm{z};\ \mathrm{Hx},\ \mathrm{R}\big)\ \mathcal{N}\big(\mathrm{x};\ \mathrm{y},\ \mathrm{P}\big)$ is a pdf, i.e. correctly normalized.

This can be obtained by a completion of the squares, i.e. by using one of the binomial formulae (vector version, proof: multiply!):

$$\boxed{(\mathrm{u}-\mathrm{v})^\top\mathrm{U}^{-1}(\mathrm{u}-\mathrm{v}) = \mathrm{u}^\top\mathrm{U}^{-1}\mathrm{u} - 2\mathrm{u}^\top\mathrm{U}^{-1}\mathrm{v} + \mathrm{v}\mathrm{U}^{-1}\mathrm{v}}$$

$$p(\mathbf{z},\mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \ \exp\Big\{ -\tfrac{1}{2}\underbrace{[(\mathbf{z}-\mathbf{H}\mathbf{x})^\top\mathbf{R}^{-1}(\mathbf{z}-\mathbf{H}\mathbf{x}) + (\mathbf{x}-\mathbf{y})^\top\mathbf{P}^{-1}(\mathbf{x}-\mathbf{y})]}_{=:(*)}\Big\}$$

abbreviations: $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$(*) = (\mathbf{A}\mathbf{u})^\top\mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + \big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)^\top\mathbf{P}^{-1}\big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)$$

$$p(\mathbf{z},\mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \ \mathsf{exp}\Big\{ -\tfrac{1}{2}\underbrace{[(\mathbf{z}-\mathbf{H}\mathbf{x})^\top\mathbf{R}^{-1}(\mathbf{z}-\mathbf{H}\mathbf{x}) + (\mathbf{x}-\mathbf{y})^\top\mathbf{P}^{-1}(\mathbf{x}-\mathbf{y})]}_{=:(*)}\Big\}$$

abbreviations: $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned}(*) &= (\mathbf{A}\mathbf{u})^\top\mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + \big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)^\top\mathbf{P}^{-1}\big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)\\ &= \mathbf{u}^\top(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A})\mathbf{u} + (\mathbf{u}-\mathbf{y}')^\top(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})(\mathbf{u}-\mathbf{y}') \qquad \text{multiply!}\end{aligned}$$

$$p(\mathbf{z},\mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \text{ exp}\Big\{-\tfrac{1}{2}\underbrace{[(\mathbf{z}-\mathbf{H}\mathbf{x})^\top\mathbf{R}^{-1}(\mathbf{z}-\mathbf{H}\mathbf{x}) + (\mathbf{x}-\mathbf{y})^\top\mathbf{P}^{-1}(\mathbf{x}-\mathbf{y})]}_{=:(*)}\Big\}$$

abbreviations: $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned}(*) &= (\mathbf{A}\mathbf{u})^\top\mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + \big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)^\top\mathbf{P}^{-1}\big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)\\ &= \mathbf{u}^\top(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A})\mathbf{u} + (\mathbf{u}-\mathbf{y}')^\top(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})(\mathbf{u}-\mathbf{y}') \qquad \text{multiply!}\\ &= \mathbf{u}^\top\underbrace{(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A} + \mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})}_{=:\mathbf{U}^{-1}}\mathbf{u} - 2\mathbf{u}^\top\underbrace{(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})\mathbf{y}'}_{=:\mathbf{U}^{-1}\mathbf{v}} + \text{const.}\end{aligned}$$

$$p(\mathbf{z},\mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \ \mathsf{exp}\Big\{-\tfrac{1}{2}\underbrace{[(\mathbf{z}-\mathbf{H}\mathbf{x})^\top\mathbf{R}^{-1}(\mathbf{z}-\mathbf{H}\mathbf{x}) + (\mathbf{x}-\mathbf{y})^\top\mathbf{P}^{-1}(\mathbf{x}-\mathbf{y})]}_{=:(*)}\Big\}$$

abbreviations: $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned}
(*) &= (\mathbf{A}\mathbf{u})^\top\mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + \big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)^\top\mathbf{P}^{-1}\big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big) \\
&= \mathbf{u}^\top(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A})\mathbf{u} + (\mathbf{u}-\mathbf{y}')^\top(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})(\mathbf{u}-\mathbf{y}') \qquad \text{multiply!} \\
&= \mathbf{u}^\top\underbrace{(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A} + \mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})}_{=:\mathbf{U}^{-1}}\mathbf{u} - 2\mathbf{u}^\top\underbrace{(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})\mathbf{y}'}_{=:\mathbf{U}^{-1}\mathbf{v}} + \text{const.} \\
&= \mathbf{u}^\top\mathbf{U}^{-1}\mathbf{u} - 2\mathbf{u}^\top\mathbf{U}^{-1}\mathbf{v} + \mathbf{v}\mathbf{U}^{-1}\mathbf{v} - \mathbf{v}\mathbf{U}^{-1}\mathbf{v} + \text{const.} \\
&= (\mathbf{u}-\mathbf{v})^\top\mathbf{U}^{-1}(\mathbf{u}-\mathbf{v}) + \text{const.'} \qquad \text{(completion of the squares!)}
\end{aligned}$$

$$p(\mathbf{z},\mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}}\ \exp\Big\{ -\tfrac{1}{2}\underbrace{[(\mathbf{z}-\mathbf{H}\mathbf{x})^\top\mathbf{R}^{-1}(\mathbf{z}-\mathbf{H}\mathbf{x}) + (\mathbf{x}-\mathbf{y})^\top\mathbf{P}^{-1}(\mathbf{x}-\mathbf{y})]}_{=:(*)}\Big\}$$

abbreviations: $\mathbf{A} = (\mathbf{I}, -\mathbf{H})$, $\mathbf{B} = (\mathbf{O}, \mathbf{I})$, $\mathbf{u} = (\mathbf{z}, \mathbf{x})$, $\mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned}
(*) &= (\mathbf{A}\mathbf{u})^\top\mathbf{R}^{-1}(\mathbf{A}\mathbf{u}) + \big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)^\top\mathbf{P}^{-1}\big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big) \\
&= \mathbf{u}^\top(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A})\mathbf{u} + (\mathbf{u}-\mathbf{y}')^\top(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})(\mathbf{u}-\mathbf{y}') \qquad \text{multiply!} \\
&= \mathbf{u}^\top\underbrace{(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A} + \mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})}_{=:\mathbf{U}^{-1}}\mathbf{u} - 2\mathbf{u}^\top\underbrace{(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})\mathbf{y}'}_{=:\mathbf{U}^{-1}\mathbf{v}} + \text{const.} \\
&= \mathbf{u}^\top\mathbf{U}^{-1}\mathbf{u} - 2\mathbf{u}^\top\mathbf{U}^{-1}\mathbf{v} + \mathbf{v}\mathbf{U}^{-1}\mathbf{v} - \mathbf{v}\mathbf{U}^{-1}\mathbf{v} + \text{const.} \\
&= (\mathbf{u}-\mathbf{v})^\top\mathbf{U}^{-1}(\mathbf{u}-\mathbf{v}) + \text{const.'} \qquad \text{(completion of the squares!)}
\end{aligned}$$

$$\mathbf{U} = [\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A} + \mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B}]^{-1} = \Big[\begin{pmatrix}\mathbf{R}^{-1} \\ -\mathbf{H}^\top\mathbf{R}^{-1}\end{pmatrix}(\mathbf{I}, -\mathbf{H}) + \begin{pmatrix}\mathbf{O} \\ \mathbf{P}^{-1}\end{pmatrix}(\mathbf{O}, \mathbf{I})\Big]^{-1}$$

$$= \begin{pmatrix}\mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{H} \\ -\mathbf{H}^\top\mathbf{R}^{-1} & \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H} + \mathbf{P}^{-1}\end{pmatrix}^{-1}$$

$$\mathbf{v} = \mathbf{U}(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})\mathbf{y}' = \mathbf{U}\begin{pmatrix}\mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{P}^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{O} \\ \mathbf{y}\end{pmatrix} = \mathbf{U}\begin{pmatrix}\mathbf{O} \\ \mathbf{P}^{-1}\mathbf{y}\end{pmatrix}$$

The inverse of a partitioned symmetric matrix is:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}$$

mit: $\mathbf{S} = \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C}$

"*Schur Complement* of the matrix $\mathbf{A}$"

proof: verify!

The inverse of a partitioned symmetric matrix is:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}$$

with: $\mathbf{S} = \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C}$

"*Schur Complement* of the matrix $\mathbf{A}$"

proof: verify!

An alternative formulation:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{T}^{-1} & -\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{C}^\top\mathbf{T}^{-1} & \mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{C}^\top\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix}$$

with: $\mathbf{T} = \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top$

proof: verify!

The inverse of a partitioned symmetric matrix is:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}$$

with: $\mathbf{S} = \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C}$

"*Schur Complement* of the matrix $\mathbf{A}$"

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An alternative formulation:

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with: $\mathbf{T} = \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top$

proof: verify!

In particular, the "matrix inversion lemma" holds:

$$(\mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}(\mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}^\top\mathbf{A}^{-1}$$

$$p(\mathbf{z},\mathbf{x}) = (|2\pi\mathbf{R}||2\pi\mathbf{P}|)^{\frac{1}{2}} \text{ exp}\Big\{ -\tfrac{1}{2}\underbrace{[(\mathbf{z}-\mathbf{Hx})^\top\mathbf{R}^{-1}(\mathbf{z}-\mathbf{Hx}) + (\mathbf{x}-\mathbf{y})^\top\mathbf{P}^{-1}(\mathbf{x}-\mathbf{y})]}_{=:(*)}\Big\}$$

abbreviations: $\mathbf{A} = (\mathbf{I}, -\mathbf{H}), \quad \mathbf{B} = (\mathbf{O}, \mathbf{I}), \quad \mathbf{u} = (\mathbf{z}, \mathbf{x}), \quad \mathbf{y}' = (\mathbf{O}, \mathbf{y})^\top$

$$\begin{aligned}
(*) &= (\mathbf{Au})^\top\mathbf{R}^{-1}(\mathbf{Au}) + \big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big)^\top\mathbf{P}^{-1}\big(\mathbf{B}(\mathbf{u}-\mathbf{y}')\big) \\
&= \mathbf{u}^\top(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A})\mathbf{u} + (\mathbf{u}-\mathbf{y}')^\top(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})(\mathbf{u}-\mathbf{y}') \qquad \text{multiply!} \\
&= \mathbf{u}^\top\underbrace{(\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A} + \mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})}_{=:\mathbf{U}^{-1}}\mathbf{u} - 2\mathbf{u}^\top\underbrace{(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})\mathbf{y}'}_{=:\mathbf{U}^{-1}\mathbf{v}} + \text{const.} \\
&= \mathbf{u}^\top\mathbf{U}^{-1}\mathbf{u} - 2\mathbf{u}^\top\mathbf{U}^{-1}\mathbf{v} + \mathbf{v}\mathbf{U}^{-1}\mathbf{v} - \mathbf{v}\mathbf{U}^{-1}\mathbf{v} + \text{const.} \\
&= (\mathbf{u}-\mathbf{v})^\top\mathbf{U}^{-1}(\mathbf{u}-\mathbf{v}) + \text{const.'} \qquad \text{(completion of the squares!)}
\end{aligned}$$

$$\mathbf{U} = [\mathbf{A}^\top\mathbf{R}^{-1}\mathbf{A} + \mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B}]^{-1} = \left[\begin{pmatrix}\mathbf{R}^{-1} \\ -\mathbf{H}^\top\mathbf{R}^{-1}\end{pmatrix}(\mathbf{I}, -\mathbf{H}) + \begin{pmatrix}\mathbf{O} \\ \mathbf{P}^{-1}\end{pmatrix}(\mathbf{O}, \mathbf{I})\right]^{-1}$$

$$= \begin{pmatrix}\mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{H} \\ -\mathbf{H}^\top\mathbf{R}^{-1} & \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H} + \mathbf{P}^{-1}\end{pmatrix}^{-1} = \begin{pmatrix}\mathbf{R} + \mathbf{HPH}^\top & \mathbf{HP} \\ \mathbf{PH}^\top & \mathbf{P}\end{pmatrix} = \begin{pmatrix}\mathbf{S} & \mathbf{HP} \\ \mathbf{PH}^\top & \mathbf{P}\end{pmatrix}$$

$$\mathbf{v} = \mathbf{U}(\mathbf{B}^\top\mathbf{P}^{-1}\mathbf{B})\mathbf{y}' = \mathbf{U}\begin{pmatrix}\mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{P}^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{O} \\ \mathbf{y}\end{pmatrix} = \mathbf{U}\begin{pmatrix}\mathbf{O} \\ \mathbf{P}^{-1}\mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{Hy} \\ \mathbf{y}\end{pmatrix}$$

therefore: $$p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\left(\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix}; \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}\right)$$ (constants: normalize!)

$$\mathcal{N}\big(\mathbf{z};\ \mathbf{H}\mathbf{x},\ \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{P}\big)$$

$$= \underbrace{\mathcal{N}\big(\mathbf{z};\ \mathbf{H}\mathbf{y},\ \mathbf{S}\big)}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}\big(\mathbf{x};\ \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\ \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top\big) \\ \mathcal{N}\big(\mathbf{x};\ \mathbf{Q}(\mathbf{P}^{-1}\mathbf{y} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\ \mathbf{Q}\big) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}$$

$$\mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

- Interpret $\mathcal{N}\big(\mathbf{z};\ \mathbf{H}\mathbf{x},\ \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{P}\big)$ as a joint density: $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x})\,p(\mathbf{x})$! ✓

  - Show that $p(\mathbf{z}, \mathbf{x})$ is a Gaussian: $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\Big(\big(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\big);\ \big(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\big),\ \big(\begin{smallmatrix}\mathbf{S} & \mathbf{H}\mathbf{P}\\ \mathbf{P}\mathbf{H}^\top & \mathbf{P}\end{smallmatrix}\big)\Big)$! ✓

  - Calculate from $p(\mathbf{z}, \mathbf{x})$ the marginal and conditional densities $p(\mathbf{z})$, $p(\mathbf{x}|\mathbf{z})$!

$$p(\mathbf{z}) = \textstyle\int d\mathbf{x}\; p(\mathbf{z}, \mathbf{x}) \qquad \text{and} \qquad p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z},\mathbf{x})}{p(\mathbf{z})}$$

  - Therefore: $p(\mathbf{z}|\mathbf{x})\,p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})\,p(\mathbf{z})$!

$$p(\mathbf{z}) = \int d\mathbf{x}\, p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x}\ \mathsf{exp}\{-\tfrac{1}{2} \underbrace{\left[\left(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\right) - \left(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\right)\right]^\top \left(\begin{smallmatrix}\mathbf{S} & \mathbf{H}\mathbf{P}\\ \mathbf{P}\mathbf{H}^\top & \mathbf{P}\end{smallmatrix}\right)^{-1} \left[\left(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\right) - \left(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\right)\right]}_{=:(*)}\}$$

Idea: Separate the integration variable $\mathbf{z}$ in a quadratic form!

$$p(\mathbf{z}) = \int d\mathbf{x}\, p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x}\ \exp\Big\{-\tfrac{1}{2} \underbrace{\Big[\Big(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\Big) - \Big(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\Big)\Big]^\top \Big(\begin{smallmatrix}\mathbf{S} & \mathbf{H}\mathbf{P}\\ \mathbf{P}\mathbf{H}^\top & \mathbf{P}\end{smallmatrix}\Big)^{-1} \Big[\Big(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\Big) - \Big(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\Big)\Big]}_{=:(*)}\Big\}$$

Idea: Separate the integration variable $\mathbf{z}$ in a quadratic form!

$$(*) = \Big(\begin{smallmatrix}\xi\\ \eta\end{smallmatrix}\Big)^\top \Big(\begin{smallmatrix}\mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{H}\\ -\mathbf{H}^\top\mathbf{R}^{-1} & \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}+\mathbf{P}^{-1}\end{smallmatrix}\Big) \Big(\begin{smallmatrix}\xi\\ \eta\end{smallmatrix}\Big) \qquad \text{mit:} \quad \Big(\begin{smallmatrix}\xi\\ \eta\end{smallmatrix}\Big) = \Big(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\Big) - \Big(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\Big)$$

$$p(\mathbf{z}) = \int d\mathbf{x}\, p(\mathbf{z},\mathbf{x}) \propto \int d\mathbf{x}\ \exp\{-\tfrac{1}{2} \underbrace{\left[\begin{pmatrix}\mathbf{z}\\ \mathbf{x}\end{pmatrix} - \begin{pmatrix}\mathbf{Hy}\\ \mathbf{y}\end{pmatrix}\right]^\top \begin{pmatrix}\mathbf{S} & \mathbf{HP}\\ \mathbf{PH}^\top & \mathbf{P}\end{pmatrix}^{-1} \left[\begin{pmatrix}\mathbf{z}\\ \mathbf{x}\end{pmatrix} - \begin{pmatrix}\mathbf{Hy}\\ \mathbf{y}\end{pmatrix}\right]}_{=:(*)}\}$$

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$$(*) = \begin{pmatrix}\xi\\ \eta\end{pmatrix}^\top \begin{pmatrix}\mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{H}\\ -\mathbf{H}^\top\mathbf{R}^{-1} & \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}+\mathbf{P}^{-1}\end{pmatrix} \begin{pmatrix}\xi\\ \eta\end{pmatrix} \qquad \text{mit:} \quad \begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\mathbf{z}\\ \mathbf{x}\end{pmatrix} - \begin{pmatrix}\mathbf{Hy}\\ \mathbf{y}\end{pmatrix}$$

$$= \xi^\top\mathbf{R}^{-1}\xi - 2\eta^\top \underbrace{\mathbf{H}^\top\mathbf{R}^{-1}\xi}_{=:\mathbf{Q}^{-1}\gamma} + \eta^\top \underbrace{(\mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H} + \mathbf{P}^{-1})}_{=:\mathbf{Q}^{-1}} \eta$$

$$p(\mathbf{z}) = \int d\mathbf{x}\, p(\mathbf{z}, \mathbf{x}) \propto \int d\mathbf{x}\ \exp\Big\{-\frac{1}{2} \underbrace{\Big[\binom{\mathbf{z}}{\mathbf{x}} - \binom{\mathbf{H}\mathbf{y}}{\mathbf{y}}\Big]^\top \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}^{-1} \Big[\binom{\mathbf{z}}{\mathbf{x}} - \binom{\mathbf{H}\mathbf{y}}{\mathbf{y}}\Big]}_{=:(*)}\Big\}$$

Idea: Separate the integration variable $\mathbf{z}$ in a quadratic form!

$$(*) = \binom{\xi}{\eta}^\top \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{H} \\ -\mathbf{H}^\top\mathbf{R}^{-1} & \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}+\mathbf{P}^{-1} \end{pmatrix} \binom{\xi}{\eta} \qquad \text{mit:} \quad \binom{\xi}{\eta} = \binom{\mathbf{z}}{\mathbf{x}} - \binom{\mathbf{H}\mathbf{y}}{\mathbf{y}}$$

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$$= \xi^\top\mathbf{R}^{-1}\xi - \gamma^\top\mathbf{Q}^{-1}\gamma + (\eta - \gamma)^\top\mathbf{Q}^{-1}(\eta - \gamma) \qquad \text{binomische Formel, quadr.Erg.!}$$

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$$= \xi^\top\mathbf{R}^{-1}\xi - \xi^\top(\mathbf{R}^{-1}\mathbf{H}\mathbf{Q}\mathbf{H}^\top\mathbf{R}^{-1})\xi + (\eta-\gamma)^\top\mathbf{Q}^{-1}(\eta-\gamma), \quad \gamma = \mathbf{Q}\mathbf{H}^\top\mathbf{R}^{-1}$$

$$= \xi^\top(\mathbf{R}^{-1} - \mathbf{P}^{-1}\mathbf{H}\mathbf{Q}\mathbf{H}^\top\mathbf{R}^{-1})\xi + (\eta-\gamma)^\top\mathbf{Q}^{-1}(\eta-\gamma)$$

$$p(\mathbf{z}) = \int d\mathbf{x}\, p(\mathbf{z},\mathbf{x}) \propto \int d\mathbf{x}\ \exp\{-\tfrac{1}{2} \underbrace{\left[\begin{pmatrix}\mathbf{z}\\\mathbf{x}\end{pmatrix} - \begin{pmatrix}\mathbf{Hy}\\\mathbf{y}\end{pmatrix}\right]^\top \begin{pmatrix}\mathbf{S} & \mathbf{HP}\\ \mathbf{PH}^\top & \mathbf{P}\end{pmatrix}^{-1} \left[\begin{pmatrix}\mathbf{z}\\\mathbf{x}\end{pmatrix} - \begin{pmatrix}\mathbf{Hy}\\\mathbf{y}\end{pmatrix}\right]}_{=:(*)}\}$$

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$$= \xi^\top(\mathbf{R}^{-1} - \mathbf{P}^{-1}\mathbf{HQH}^\top\mathbf{R}^{-1})\xi + (\eta-\gamma)^\top\mathbf{Q}^{-1}(\eta-\gamma)$$

$$= \xi^\top(\mathbf{R} + \mathbf{HPH}^\top)^{-1}\xi + (\eta-\gamma)^\top\mathbf{Q}^{-1}(\eta-\gamma) \qquad \text{Inversionslemma!}$$

$$= \xi^\top\mathbf{S}^{-1}\xi + (\eta-\gamma)^\top\mathbf{Q}^{-1}(\eta-\gamma)$$

$$p(\mathbf{z}) = \int d\mathbf{x}\, p(\mathbf{x}, \mathbf{z})$$

$$\propto \exp[-\tfrac{1}{2}(\xi^\top \mathbf{S}^{-1}\xi)] \int d\mathbf{y}\ \exp[-\tfrac{1}{2}(\eta - \boldsymbol{\gamma})^\top \mathbf{Q}^{-1}(\eta - \boldsymbol{\gamma})]$$

$$\propto \exp[-\tfrac{1}{2}(\mathbf{z} - \mathbf{H}\mathbf{y})^\top \mathbf{S}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{y})]$$

therefore: $$\boxed{p(\mathbf{x}) = \mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}\big)}$$

(constants result from: $\int d\mathbf{x}\, p(\mathbf{x}) = 1$)

$$p(\mathbf{x}|\mathbf{z}) = \frac{p(\mathbf{z}, \mathbf{x})}{p(\mathbf{z})} \propto \text{exp}[-\tfrac{1}{2}(\eta - \gamma)^\top \mathbf{Q}^{-1}(\eta - \gamma)]$$

$$\mathbf{Q} = \underbrace{(\mathbf{H}^\top \mathbf{R}^{-1}\mathbf{H} + \mathbf{P}^{-1})^{-1}}_{\text{2. Version!}} = \mathbf{P} - \mathbf{P}\mathbf{H}^\top \mathbf{S}^{-1}\mathbf{H}\mathbf{P} \qquad \text{inversion lemma!}$$

$$= \underbrace{\mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{H}^\top}_{\text{1st version!}} = \underbrace{(\mathbf{I} - \mathbf{W}\mathbf{H})\mathbf{P}}_{\text{3rd version!}}, \qquad \text{mit:} \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top \mathbf{S}^{-1}$$

$$\begin{aligned}
\gamma &= \mathbf{Q}\mathbf{H}^\top \mathbf{R}^{-1}\xi = (\mathbf{I} - \mathbf{W}\mathbf{H})\mathbf{P}\mathbf{H}^\top \mathbf{R}^{-1}(\mathbf{z} - \mathbf{H}\mathbf{y}) \\
&= (\mathbf{P}\mathbf{H}^\top \mathbf{R}^{-1} - \mathbf{W}\mathbf{H}\mathbf{P}\mathbf{H}^\top \mathbf{R}^{-1})(\mathbf{z} - \mathbf{H}\mathbf{y}) \\
&= \mathbf{P}\mathbf{H}^\top(\mathbf{R}^{-1} - \mathbf{S}^{-1}(\mathbf{S} - \mathbf{R})\mathbf{R}^{-1})(\mathbf{z} - \mathbf{H}\mathbf{y}) = \mathbf{W}(\mathbf{z} - \mathbf{H}\mathbf{y}) \\
&= \mathbf{Q}(\mathbf{P}^{-1}\mathbf{y} + \mathbf{H}^\top \mathbf{R}^{-1}\mathbf{z}) \qquad \text{2nd version, inversion lemma!}
\end{aligned}$$

therefore: $$\boxed{p(\mathbf{x}|\mathbf{z}) = \mathcal{N}\big(\mathbf{x};\, \mathbf{q},\, \mathbf{Q}\big)}$$

$$\mathbf{q} = \mathbf{y} + \mathbf{W}(\mathbf{z} - \mathbf{H}\mathbf{y}), \qquad \mathbf{Q} = \mathbf{Y} - \mathbf{W}\mathbf{S}\mathbf{W}^\top$$

(constants result from: $\int d\mathbf{x}\, p(\mathbf{x}|\mathbf{z}) = 1$)