Kalman filter: linear Gaussian likelihood/dynamics, $x_k = (r_k^T, \dot{r}_k^T, \ddot{r}_k^T)^T$, $\mathcal{Z}^k = \{z_k, \mathcal{Z}^{k-1}\}$

**initiation:** $p(x_0) = \mathcal{N}(x_0; x_{0|0}, P_{0|0})$, initial ignorance: $P_{0|0}$ ‘large’

**prediction:** $\mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \xrightarrow{\text{dynamics model}} \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1})$

\[
x_{k|k-1} = F_{k|k-1} x_{k-1|k-1}
\]

\[
\begin{align*}
P_{k|k-1} &= F_{k|k-1} P_{k-1|k-1} F_{k|k-1}^\top + D_{k|k-1} \\
\end{align*}
\]

**filtering:** $\mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \xrightarrow{\text{current measurement } z_k} \mathcal{N}(x_k; x_{k|k}, P_{k|k})$

\[
\begin{align*}
x_k &= x_{k|k-1} + W_{k|k-1} \nu_{k|k-1} \\
P_{k|k} &= P_{k|k-1} - W_{k|k-1} S_{k|k-1} W_{k|k-1}^\top \\
W_{k|k-1} &= P_{k|k-1} H_{k}^\top S_{k|k-1}^{-1}
\end{align*}
\]

\[
\begin{align*}
\nu_{k|k-1} &= z_k - H_k x_{k|k-1} \\
S_{k|k-1} &= H_k P_{k|k-1} H_k^\top + R_k
\end{align*}
\]

‘KALMAN gain matrix’

**retrodiction:** $\mathcal{N}(x_l; x_{l|k}, P_{l|k}) \xleftarrow{\text{filtering, prediction}} \mathcal{N}(x_{l+1}; x_{l+1|k}, P_{l+1|k})$

\[
\begin{align*}
x_{l|k} &= x_{l|l} + W_{l|l+1}(x_{l+1|k} - x_{l+1|l}) \\
P_{l|k} &= P_{l|l} + W_{l|l+1}(P_{l+1|k} - P_{l+1|l}) W_{l|l+1}^\top
\end{align*}
\]

\[
\begin{align*}
W_{l|l+1} &= P_{l|l} F_{l+1|l}^\top P_{l+1|l}^{-1}
\end{align*}
\]
Continuous Time Retrodiction for $t_l < t_l + \theta < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(x_l | Z^k)$ and $p(x_{l+1} | Z^k)$ based on the evolution model:

$$p(x_{l+\theta} | Z^k) = \int dx_{l+1} \ p(x_{l+\theta}, x_{l+1} | Z^k)$$

$$= \int dx_{l+1} \ p(x_{l+\theta} | x_{l+1}, Z^k) \ p(x_{l+1} | Z^k)$$
Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(x_l|Z^k)$ and $p(x_{l+1}|Z^k)$ based on the evolution model:

$$p(x_{l+\theta}|Z^k) = \int dx_{l+1} \ p(x_{l+\theta}, x_{l+1}|Z^k)$$
$$= \int dx_{l+1} \ p(x_{l+\theta}|x_{l+1}, Z^k) \ p(x_{l+1}|Z^k)$$

where:

$$p(x_{l+\theta}|x_{l+1}, Z^k) = \frac{p(x_{l+1}|x_{l+\theta}) \ p(x_{l+\theta}|Z^l)}{\int dx_{l+\theta} \ p(x_{l+1}|x_{l+\theta}) \ p(x_{l+\theta}|Z^l)}$$

with:

$$p(x_{l+1}|x_{l+\theta}) = \mathcal{N}(x_{l+1}; F_{l+1|l+\theta} x_{l+\theta}, D_{l+1|l+\theta})$$

$$p(x_{l+\theta}|Z^l) = \int dx_{l} \ p(x_{l+\theta}|x_{l}) \ p(x_{l}|Z^l)$$

$$p(x_{l+1}|Z^l) = \int dx_{l+\theta} \ p(x_{l+1}|x_{l+\theta}) \ p(x_{l+\theta}|Z^l)$$
$$= \mathcal{N}(x_{l+1}; x_{l+1|l}, P_{l+1|l})$$
Continuous Time Retrodiction for $t_l < t_l + \theta < t_l + 1$ with $0 < \theta < 1$

Interpolate between $p(x_l|Z^k)$ and $p(x_{l+1}|Z^k)$ based on the evolution model:

$$
p(x_{l+\theta}|Z^k) = \int dx_{l+1} p(x_{l+\theta}, x_{l+1}|Z^k) = \int dx_{l+1} p(x_{l+\theta}|x_{l+1}, Z^k) p(x_{l+1}|Z^k)
$$

where:

$$
p(x_{l+\theta}|x_{l+1}, Z^k) = \frac{p(x_{l+1}|x_{l+\theta}) p(x_{l+\theta}|Z^l)}{\int dx_{l+\theta} p(x_{l+1}|x_{l+\theta}) p(x_{l+\theta}|Z^l)}
$$

with:

$$
p(x_{l+1}|x_{l+\theta}) = \mathcal{N}(x_{l+1}; F_{l+1|l+\theta} x_{l+\theta}, D_{l+1|l+\theta})$$

$$
p(x_{l+\theta}|Z^l) = \int dx_l p(x_{l+\theta}|x_l) p(x_l|Z^l)$$

$$
p(x_{l+1}|Z^l) = \int dx_{l+\theta} p(x_{l+1}|x_{l+\theta}) p(x_{l+\theta}|Z^l) = \mathcal{N}(x_{l+1}; x_{l+1|l}, P_{l+1|l})$$

Looks like a Kalman filtering update!
\[
p(x_{l+\theta} | x_{l+1}, Z^k) \propto p(x_{l+1} | x_{l+\theta}) \, p(x_{l+\theta} | Z^l)
\]
Looks like filtering!

\[
p(x_{l+ \theta} | Z^k) = \int dx_{l+1} \, p(x_{l+\theta} | x_{l+1}, Z^k) \, p(x_{l+1} | Z^k)
\]
Looks like prediction!
\[
p(x_{l+\theta}|x_{l+1}, Z^k) \propto p(x_{l+1}|x_{l+\theta}) \ p(x_{l+\theta}|Z^l) \\
= \mathcal{N}(x_{l+\theta}; a_{l+\theta|l+1}, \Delta_{l+\theta|l+1})
\]

\[
a_{l+\theta|l+1} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}(x_{l+1} - F_{l+\theta|l+1} x_{l+\theta|l}) \\
= x_{l+\theta|l} - \Phi_{l+\theta|l+1} x_{l+1} + \Phi_{l+\theta|l+1} x_{l+1}
\]

\[
\Delta_{l+\theta|l+1} = P_{l+\theta|l} - \Phi_{l+\theta|l+1} P_{l+1|l} \Phi_{l+\theta|l+1}^T
\]

\[
\Phi_{l+\theta|l+1} = P_{l+\theta|l} F_{l+1|l+\theta}^{-1} P_{l+1|l} F_{l+1|l+\theta}^T
\]

\[
P_{l+1|l} = F_{l+1|l+\theta} P_{l+\theta|l} F_{l+1|l+\theta}^T + D_{l+1|l+\theta}.
\]

\[
p(x_{l+\theta}|Z^k) = \int dx_{l+1} \ p(x_{l+\theta}|x_{l+1}, Z^k) \ p(x_{l+1}|Z^k) \\
\text{Looks like prediction!}
\]
\[ p(x_{l+\theta}|x_{l+1}, \mathcal{Z}^k) \propto p(x_{l+1}|x_{l+\theta}) p(x_{l+\theta}|\mathcal{Z}^l) \]  
Looks like filtering!

\[ = \mathcal{N}(x_{l+\theta}; a_{l+\theta|l+1}, \Delta_{l+\theta|l+1}) \]

\[ = \mathcal{N}(b_{l+\theta|l+1}; \Phi_{l+\theta|l+1} x_{l+1}, \Delta_{l+\theta|l+1}) \]

\[ a_{l+\theta|l+1} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}(x_{l+1} - F_{l+1|l+\theta} x_{l+1}|l) \]
\[ = x_{l+\theta|l} - \Phi_{l+\theta|l+1} x_{l+1}|l + \Phi_{l+\theta|l+1} x_{l+1} \]
\[ b_{l+\theta|l+1} = x_{l+\theta} - x_{l+\theta|l} + \Phi_{l+\theta|l+1} x_{l+1}|l \]
\[ \Delta_{l+\theta|l+1} = P_{l+\theta|l} - \Phi_{l+\theta|l+1} P_{l+1|l} \Phi_{l+\theta|l+1}^\top \]
\[ \Phi_{l+\theta|l+1} = P_{l+\theta|l} F_{l+1|l+\theta}^\top P_{l+1|l}^{-1} \]
\[ P_{l+1|l} = F_{l+1|l+\theta} P_{l+\theta|l} F_{l+1|l+\theta}^\top + D_{l+1|l+\theta}. \]

\[ p(x_{l+\theta}|\mathcal{Z}^k) = \int dx_{l+1} p(x_{l+\theta}|x_{l+1}, \mathcal{Z}^k) p(x_{l+1}|\mathcal{Z}^k) \]  
Looks like prediction!
\[ p(x_{l+\theta} | x_{l+1}, Z^k) \propto p(x_{l+1} | x_{l+\theta}) \; p(x_{l+\theta} | Z^l) \quad \text{Looks like filtering!} \]

\[
= \mathcal{N}(x_{l+\theta}; a_{l+\theta|l+1}, \Delta_{l+\theta|l+1}) \\
= \mathcal{N}(b_{l+\theta|l+1}; \Phi_{l+\theta|l+1}x_{l+1}, \Delta_{l+\theta|l+1})
\]

\[
a_{l+\theta|l+1} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}(x_{l+1} - F_{l+1|l+\theta}x_{l+\theta|l}) \\
= x_{l+\theta|l} - \Phi_{l+\theta|l+1}x_{l+1} + \Phi_{l+\theta|l+1}x_{l+1}
\]

\[
b_{l+\theta|l+1} = x_{l+\theta} - x_{l+\theta|l} + \Phi_{l+\theta|l+1}x_{l+1}
\]

\[
\Delta_{l+\theta|l+1} = P_{l+\theta|l} - \Phi_{l+\theta|l+1}P_{l+1|l}\Phi_{l+\theta|l+1}^T
\]

\[
\Phi_{l+\theta|l+1} = P_{l+\theta|l}F_{l+1|l+\theta}^T P_{l+1|l}^{-1}
\]

\[
P_{l+1|l} = F_{l+1|l+\theta} P_{l+\theta|l} F_{l+1|l+\theta}^T + D_{l+1|l+\theta}.
\]

\[ p(x_{l+\theta} | Z^k) = \int dx_{l+1} \; p(x_{l+\theta} | x_{l+1}, Z^k) \; p(x_{l+1} | Z^k) \quad \text{Looks like prediction!} \]

\[
= \mathcal{N}(x_{l+\theta}; x_{l+\theta|k}, x_{l+\theta|k})
\]

\[
x_{l+\theta|k} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}(x_{l+1|k} - x_{l+1|l})
\]

\[
P_{l+\theta|k} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}\left(P_{l+1|k} - P_{l+1|l}\right)\Phi_{l+\theta|l+1}^T
\]
A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

\[
p(z_k \mid Z^{k-1}) = \int dx_k \ p(z_k, x_k \mid Z^{k-1}) = \int dx_k \ p(z_k \mid x_k) \ p(x_k \mid Z^{k-1})
\]
A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

\[
p(z_k | Z^{k-1}) = \int dx_k \ p(z_k, x_k | Z^{k-1}) = \int dx_k \ p(z_k | x_k) \ p(x_k | Z^{k-1})
\]

\[
= \int dx_k \ \mathcal{N} \left( z_k; H_k x_k, R_k \right) \mathcal{N} \left( x_k; x_{k|k-1}, P_{k|k-1} \right)
\]

likelihood: sensor model  prediction at time \( t_k \)
A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

\[
p(z_k | Z^{k-1}) = \int dx_k \ p(z_k, x_k | Z^{k-1}) = \int dx_k \ p(z_k | x_k) \ p(x_k | Z^{k-1})
\]

\[
= \int dx_k \ N(z_k; H_k x_k, R_k) \ N(x_k; x_{k|k-1}, P_{k|k-1})
\]

\[
= N(z_k; H_k x_{k|k-1}, S_{k|k-1}) \quad \text{(product formula)}
\]

innovation: \( \nu_{k|k-1} = z_k - H_k x_{k|k-1} \),

innovation covariance: \( S_{k|k-1} = H_k P_{k|k-1} H_k^\top + R_k \)

expectation gate: \( \nu_{k|k-1}^\top S_{k|k-1}^{-1} \nu_{k|k-1} \leq \lambda(P_c) \)

**Mahalanobis** ellipsoid containing \( z_k \) with certain probability \( P_c \)

Choose \( \lambda(P_c) \) (“gating parameter”) properly!

Can be looked up in a \( \chi^2 \)-table - discussed later!
Sensor data of uncertain origin

- prediction: $x_{k|k-1}, P_{k|k-1}$ (dynamics)
- innovation: $\nu_k = z_k - Hx_{k|k-1}$, white
- Mahalanobis norm: $||\nu_k||^2 = \nu_k^T S_k^{-1} \nu_k$
- expected plot: $z_k \sim N(Hx_{k|k-1}, S_k)$
- $\nu_k \sim N(0, S_k)$, $S_k = HP_{k|k-1}H^\top + R$
- gating: $||\nu_k|| < \lambda, P_c(\lambda)$ correlation prob.

missing/false plots, measurement errors, scan rate, agile targets: large gates
A Generic Tracking and Sensor Data Fusion System

Tracking & Fusion System

Sensor System

Sensing Hardware:
- Received Waveforms

Detection Process:
- Data Rate Reduction

Signal Processing:
- Parameter Estimation

A Priori Knowledge:
- Sensor Performance
- Object Characteristics
- Object Environment

Track Initiation:
- Multiple Frame
- Track Extraction

Sensor Data to Track Association

Track Maintenance:
- Prediction, Filtering
- Retrodiction

Track File Storage

Track Processing:
- Track Cancellation
- Object Classification / ID
- Track-to-Track Fusion

Man-Machine Interface:
- Object Representation
- Displaying Functions
- Interaction Facilities

Sensor Data Fusion - Methods and Applications, 5th Lecture on November 13, 2019 — slide 13
Description of the Detection Process

**Detector**: receives signals and decides on object existence

**Processor**: processes detected signals and produces measurements

‘\(D\)': detector detects an object

\(D\): object actually existent
Description of the Detection Process

**Detector:** receives signals and decides on object existence

**Processor:** processes detected signals and produces measurements

‘D’: detector detects an object  \( P_I = P(\neg D|D) \)

\( D \): object actually existent  \( P_{II} = P(D|\neg D) \)

error of 1. kind: \( P_I = P(\neg D|D) \)

error of 2. kind: \( P_{II} = P(D|\neg D) \)

measure of detection performance: \( P_D = P(D|D) \)

detector properties characterized by two parameters:

- detection probability \( P_D = 1 - P_I \)
- false alarm probability \( P_F = P_{II} \)
Description of the Detection Process

**Detector:** receives signals and decides on object existence

**Processor:** processes detected signals and produces measurements

‘\(D\)’: detector detects an object

\(D\): object actually existent

error of 1. kind: \(P_I = P(\neg 'D'|D)\)

error of 2. kind: \(P_{II} = P('D'|\neg D)\)

measure of detection performance: \(P_D = P('D'|D)\)

detector properties characterized by two parameters:

– detection probability \(P_D = 1 - P_I\)

– false alarm probability \(P_F = P_{II}\)

example (Swerling I model): \(P_D = P_D(P_F, \text{SNR}) = P_F^{1/(1+\text{SNR})}\)

detector design: Maximize detection probability \(P_D\) for a given, predefined false alarm probability \(P_F\)!
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

\[ n_k + 1 \text{ possible interpretations of the sensor data } Z_k = \{z^j_k\}_{j=1}^{n_k}! \]

- \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
- \(E_j, j = 1, \ldots, n_k\): Object detected; \(z^j_k\) is object measurement; \(n_k - 1\) false measurements

Consider the interpretations in the likelihood function \(p(Z_k, n_k | x_k)\)!
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

\(n_k + 1\) possible interpretations of the sensor data \(Z_k = \{z^j_k\}_{j=1}^{n_k}\):

- \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
- \(E_j, j = 1, \ldots, n_k\): Object detected; \(z^j_k\) is object measurement; \(n_k - 1\) false measurements

Consider the interpretations in the likelihood function \(p(Z_k, n_k|x_k)\):

\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k)
\]

\(D = \text{“object was detected”}\)
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

\(n_k + 1\) possible interpretations of the sensor data \(Z_k = \{z_k^j\}_{j=1}^{n_k}\):

- \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
- \(E_j, j = 1, \ldots, n_k\): Object detected; \(z_k^j\) is object measurement; \(n_k - 1\) false measurements

Consider the interpretations in the likelihood function \(p(Z_k, n_k|x_k)\)

\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k) \quad D = \text{“object was detected”}
\]

\[
= p(Z_k, n_k|\neg D, x_k) P(\neg D|x_k) + p(Z_k, n_k|D, x_k) P(D|x_k)
\]

\[
= p(Z_k, n_k|\neg D, x_k) \frac{1}{P_D} P(\neg D|x_k) + p(Z_k, n_k|D, x_k) P(D|x_k)
\]

sensor parameter: detection probability \(P_D\)
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

\[ n_k + 1 \text{ possible interpretations of the sensor data } Z_k = \{z^j_k\}_{j=1}^{n_k}! \]

- \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
- \(E_j, j = 1, \ldots, n_k\): Object detected; \(z^j_k\) is object measurement; \(n_k - 1\) false measurements

Consider the interpretations in the likelihood function \(p(Z_k, n_k|x_k)\)！

\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k) \quad D = \text{“object was detected”}
\]

\[
= p(Z_k, n_k, \neg D, x_k) \cdot P(\neg D|x_k) + p(Z_k, n_k, D, x_k) \cdot p(D|x_k)
\]

\[
= \frac{p(Z_k | n_k, \neg D, x_k)}{|\text{FoV}|^{n_k}} \cdot p(n_k | \neg D, x_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j | D, x_k)
\]

false measurements: Poisson distributed in #, uniformly distributed in the FoV
Modeling of False Measurements (FM)

- Probability of having $n$ FM: $p_F(n)$

  mean number of FM in the ‘Field of View’ (FoV) of a sensor:

  $$\bar{n} = \rho_F |\text{FoV}|,$$

  false measurement density $\rho_F$ (perhaps not constant)
Modeling of False Measurements (FM)

- Probability of having $n$ FM: $p_F(n)$
  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    \[ \bar{n} = \rho_F|\text{FoV}|, \quad \text{false measurement density } \rho_F \text{ (perhaps not constant)} \]
  - assumption: $n$ is a Poisson distributed RV with
    \[ p_F(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}} \]
Modeling of False Measurements (FM)

- Probability of having $n$ FM: $p_F(n)$
  
  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    
    $\bar{n} = \rho_F|\text{FoV}|$, false measurement density $\rho_F$ (perhaps not constant)
  
  - assumption: $n$ is a Poisson distributed RV with
    
    $p_F(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}$

  expectation: $\mathbb{E}[n] = \bar{n}$, variance: $\mathbb{V}[n] = \bar{n}$
normalization: \[ \sum_{n=0}^{\infty} p_F(n) = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = e^{-\bar{n}} e^{\bar{n}} = 1 \]

expectation: \[ E[n] = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{n \bar{n}^n}{n!} = e^{-\bar{n}} \sum_{n=1}^{\infty} \frac{n \bar{n}^n}{n!} = \bar{n} e^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^{n-1}}{(n-1)!} = \bar{n} \]

variance: \[ \nabla[n] = E[(n - \bar{n})^2] = E[n^2] - \bar{n}^2 = \ldots \text{exercise!} \ldots = \bar{n} \]
Modeling of False Measurements (FM)

• Probability of having \( n \) FM: \( p_F(n) \)
  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    \[ \tilde{n} = \rho_F|\text{FoV}|, \quad \text{false measurement density } \rho_F \text{ (perhaps not constant)} \]
  - assumption: \( n \) is a Poisson distributed RV with
    \[ p_F(n) = \frac{\tilde{n}^n}{n!} e^{-\tilde{n}} \]
equation
  - expectation: \( \mathbb{E}[n] = \tilde{n} \), variance: \( \mathbb{V}[n] = \tilde{n} \)

• Distribution of FM in the Field of View:
  \( p(z_1^f, \ldots, z_n^f|\text{FoV}) \)
  - FM mutually independent:
    \[ p(z_1^f, \ldots, z_n^f|\text{FoV}) = \prod_{i=1}^{n} p(z_i^f|\text{FoV}) \]
Modeling of False Measurements (FM)

- **Probability of having** \( n \) **FM:** \( p_F(n) \)

  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    \[
    \bar{n} = \rho_{F|\text{FoV}}, \quad \text{false measurement density } \rho_F \text{ (perhaps not constant)}
    \]
  - assumption: \( n \) is a Poisson distributed RV with
    \[
    p_F(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}
    \]
    expectation: \( \mathbb{E}[n] = \bar{n} \), variance: \( \mathbb{V}[n] = \bar{n} \)

- **Distribution of FM in the Field of View:** \( p(z_1^f, \ldots, z_n^f|\text{FoV}) \)

  - FM mutually independent:
    \[
    p(z_1^f, \ldots, z_n^f|\text{FoV}) = \prod_{i=1}^{n} p(z_i^f|\text{FoV})
    \]
  - uniformly distributed in the FoV:
    \[
    p(z_i^f|\text{FoV}) = |\text{FoV}|^{-1} \quad \text{(often!)}
    \]
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

\(n_k + 1\) possible interpretations of the sensor data \(Z_k = \{z^j_k\}_{j=1}^{n_k}\):

- \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
- \(E_j, j = 1, \ldots, n_k\): Object detected; \(z^j_k\) is object measurement; \(n_k - 1\) false measurements

Consider the interpretations in the likelihood function \(p(Z_k, n_k|x_k)\):

\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k)
\]

\(D = \) “object was detected”

\[
= p(Z_k, n_k, \neg D, x_k) P(\neg D|x_k) + p(Z_k, n_k, D, x_k) p(D|x_k)
\]

\[
= p(Z_k|n_k, \neg D, x_k) p(n_k|\neg D, x_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, x_k)
\]

\[
= |\text{FoV}|^{-n_k} p_F(n_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} \frac{p(Z_k|n_k, j, D, x_k) p(j|n_k, D) p(n_k|D)}{|\text{FoV}|^{-(n_k-1)} N(z^j_n, Hx_k, R)} = 1/n_k = p_F(n_k - 1)
\]

Insert Poisson distribution:

\[
p_F(n_k) = \frac{(\rho_F|\text{FoV}|^{-n_k} e^{-\rho_F|\text{FoV}|}}{n_k!}
\]
ambiguous sensor data \( (P_D < 1, \rho_F > 0) \)

\[ n_k + 1 \text{ possible interpretations of the sensor data } Z_k = \{z^j_k\}_{j=1}^{n_k}! \]

- \( E_0 \): the object was not detected; \( n_k \) false data in the Field of View (FoV)
- \( E_j, j = 1, \ldots, n_k \): Object detected; \( z^j_k \) is object measurement; \( n_k - 1 \) false measurements

Consider the interpretations in the likelihood function \( p(Z_k, n_k|x_k)! \)

\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k) \quad D = "\text{object was detected}" \\
= p(Z_k, n_k, \neg D, x_k) P(\neg D|x_k) + p(Z_k, n_k|D, x_k) p(D|x_k) \\
= p(Z_k|n_k, \neg D, x_k) p(n_k|\neg D, x_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, x_k) \\
= |\text{FoV}|^{-n_k} p_F(n_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k|n_k, j, D, x_k) p(j|n_k, D) p(n_k|D) \\
= e^{-\rho_F|\text{FoV}|} \frac{\rho_F^{-n_k}}{n_k!} \left( (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}(z^j_k; Hx_k, R) \right) \\
\]

Sensor Data Fusion - Methods and Applications, 5th Lecture on November 13, 2019 — slide 28
Likelihood Functions

The likelihood function answers the question:
What does the sensor tell about the state $x$ of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1, \rho_F = 0$

  at each time one measurement:
  
  $p(z_k|x_k) = \mathcal{N}(z_k; Hx_k, R)$

- **real conditions, one object:** $P_D < 1, \rho_F > 0$

  at each time $n_k$ measurements $Z_k = \{z_{1k}, \ldots, z_{nk}\}$

  
  $p(Z_k, n_k|x_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}(z_{jk}; Hx_k, R)$
PDAF Filter: formally analogous to Kalman Filter

Filtering (scan $k - 1$): $p(x_{k-1} | Z^{k-1}) = \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1})$ (→ initiation)

Prediction (scan $k$): $p(x_k | Z^{k-1}) \approx \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1})$ (like Kalman)

Filtering (scan $k$): $p(x_k | Z^k) \approx \sum_{j=0}^{m_k} p^j_k \mathcal{N}(x_k; x_{j|k}, P_{j|k}) \approx \mathcal{N}(x_k; x_{k|k}, P_{k|k})$
PDAF Filter: formally analogous to Kalman Filter

Filtering (scan \( k-1 \)):
\[
p(x_{k-1} | Z^{k-1}) = \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \quad (\rightarrow \text{initiation})
\]

Prediction (scan \( k \)):
\[
p(x_k | Z^{k-1}) \approx \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \quad (\text{like Kalman})
\]

Filtering (scan \( k \)):
\[
p(x_k | Z^k) \approx \sum_{j=0}^{m_k} p_j^k \mathcal{N}(x_k; x_{j|k}, P_{j|k})
\]

\[
x_{j|k} = \begin{cases} x_{k|k-1} & j=0 \\ x_{k|k-1} + W_k \nu_{j}^k & j \neq 0 \end{cases}
\]

\[
P_j^k = \begin{cases} P_{k|k-1} & j=0 \\ P_{k|k-1} - W_k S_k W_k^T & j \neq 0 \end{cases}
\]

\[
\nu_j^k = z_j^k - H x_k, \quad W_k = P_{k|k-1} H^T S_k^{-1}, \quad S_k = HP_{k|k-1} H^T + R_k
\]

\[
p_j^k = \frac{p_j^k*}{\sum_j p_j^k*} \quad p_j^k* = \begin{cases} (1 - P_D) \rho_F & j=0 \\ \frac{P_D}{\sqrt{|2\pi S_{\nu_k}|}} e^{-\frac{1}{2} \nu_{j}^T S_{\nu_k} \nu_{j}} & j \neq 0 \end{cases}
\]
**Moment Matching:** Approximate an arbitrary pdf $p(x)$ with $\mathbb{E}[x] = x$, $\mathbb{C}[x] = P$ by $p(x) \approx \mathcal{N}(x; x, P)$!

here especially: $p(x) = \sum_H p_H \mathcal{N}(x; x_H, P_H)$ (normal mixtures)

$$x = \sum_H p_H x_H$$

spread term

$$P = \sum_H p_H \left\{ P_H + (x_H - x)(x_H - x)^\top \right\}$$
Second-order Approximation of the Mixture Density:

\[
\sum_{j=1}^{m_k} p_k^j \mathcal{N}(x_k; x_{k|k}^j, P_{k|k}^j) \approx \mathcal{N}(x_k; x_{k|k}, P_{k|k})
\]

mit:

\[
x_{k|k} = \sum_{j=0}^{m_k} p_k^j x_{k|k}^j
\]

\[
P_{k|k} = \sum_{j=0}^{m_k} p_k^j \left( P_{k|k}^j + (x_{k|k}^j - x_{k|k})(x_{k|k}^j - x_{k|k})^\top \right)
\]
\[
x_{k|k} = \sum_{j=0}^{m_k} p_j x_{k|k}^j, \quad x_{0}^{k} = x_{k|k-1}, \quad x_{j}^{k} = x_{k|k-1} + W_k \nu_j^k
\]

\[
P_{k|k} = \sum_{j=0}^{m_k} p_j (P_{k|k}^j + (x_{j}^{k} - x_{k|k})(x_{j}^{k} - x_{k|k})^\top)
\]
\[ x_k|k = \sum_{j=0}^{m_k} p^j_k x^j_{k|k} \]

\[ = p^0_k x_{k|k-1} + \sum_{j=1}^{m_k} p^j_k (x_{k|k-1} + W_k \nu^j_k) \]

\[ P_{k|k} = \sum_{j=0}^{m_k} p^j_k (P^j_{k|k} + (x^j_{k|k} - x_{k|k})(x^j_{k|k} - x_{k|k})^\top) \]
\[ \mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p^j_k \mathbf{x}^j_{k|k} \]

\[ = p^0_k \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p^j_k \left( \mathbf{x}_{k|k-1} + \mathbf{W}_k \mathbf{\nu}^j_k \right) \]

\[ = \mathbf{x}_{k|k-1} \left( p^0_k + \sum_{j=1}^{m_k} p^j_k \right) + \mathbf{W}_k \sum_{j=1}^{m_k} p^j_k \mathbf{\nu}^j_k \]

\[ \text{mean!} \]

\[ \mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p^j_k \left( \mathbf{P}^j_{k|k} + \left( \mathbf{x}^j_{k|k} - \mathbf{x}_{k|k} \right) \left( \mathbf{x}^j_{k|k} - \mathbf{x}_{k|k} \right)^\top \right) \]
\[
\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p^j_k \mathbf{x}^j_{k|k}
\]

\[
= p^0_k \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p^j_k (\mathbf{x}_{k|k-1} + \mathbf{W}_k \nu^j_k) = \mathbf{x}_{k|k-1} + \mathbf{W}_k \nu_k
\]

\[
\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p^j_k (\mathbf{P}^j_{k|k} + (\mathbf{x}^j_{k|k} - \mathbf{x}_{k|k})(\mathbf{x}^j_{k|k} - \mathbf{x}_{k|k})^\top)
\]

\[\text{Combined Innovation: } \nu_k = \sum_{j=1}^{m_k} p^j_k \nu^j_k\]
\[ x_{k|k} = \sum_{j=0}^{m_k} p^j_k x^j_{k|k} \]

\[ = p^0_k x_{k|k-1} + \sum_{j=1}^{m_k} p^j_k (x_{k|k-1} + W_k \nu^j_k) = x_{k|k-1} + W_k \nu_k \]

\[ P_{k|k} = \sum_{j=0}^{m_k} p^j_k (P^j_{k|k} + (x^j_{k|k} - x_{k|k})(x^j_{k|k} - x_{k|k})^\top), \quad P^0_{k|k} = P_{k|k-1}, \quad P^j_{k|k} = P_{k|k-1} - W_k S_k W_k^\top \]

\[ = P_{k|k-1} - \sum_{j=1}^{m_k} p^j_k W_k S_k W_k^\top + \sum_{j=1}^{m_k} p^j_k W_k (\nu^j_k - \nu_k)(\nu^j_k - \nu_k)^\top W_k^\top \]

**Combined Innovation:** \[ \nu_k = \sum_{j=1}^{m_k} p^j_k \nu^j_k \]
\[ x_{k|k} = \sum_{j=0}^{m_k} p_{k}^{j} x_{k|k}^{j} \]

\[ = p_{k}^{0} x_{k|k-1} + \sum_{j=1}^{m_k} p_{k}^{j} (x_{k|k-1} + W_k \nu_k^{j}) = x_{k|k-1} + W_k \nu_k \]

\[ P_{k|k} = \sum_{j=0}^{m_k} p_{k}^{j} (P_{k|k}^{j} + (x_{k|k}^{j} - x_{k|k}) (x_{k|k}^{j} - x_{k|k})^\top) \]

\[ = P_{k|k-1} - \sum_{j=1}^{m_k} p_{k}^{j} W_k S_k W_k^\top + \sum_{j=1}^{m_k} p_{k}^{j} W_k (\nu_k^{j} - \nu_k) (\nu_k^{j} - \nu_k)^\top W_k^\top \]

\[ = P_{k|k-1} - (1 - p_{k}^{0}) W_k S_k W_k^\top + W_k \left[ \sum_{j=1}^{m_k} p_{k}^{j} \nu_k^{j} \nu_k^{j\top} - \nu_k \nu_k^\top \right] W_k^\top \]

**Combined Innovation:** \[ \nu_k = \sum_{j=1}^{m_k} p_{k}^{j} \nu_k^{j} \]
**PDAF Filter: formally analog to Kalman Filter**

**Filtering (scan \( k-1 \)):**

\[
p(x_{k-1} | \mathcal{Z}^{k-1}) = \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \quad (\rightarrow \text{initiation})
\]

**prediction (scan \( k \)):**

\[
p(x_k | \mathcal{Z}^{k-1}) \approx \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \quad (\text{like Kalman})
\]

**Filtering (scan \( k \)):**

\[
p(x_k | \mathcal{Z}^k) \approx \sum_{j=0}^{m_k} p_k^j \mathcal{N}(x_k; x_{k|k}^j, P_{k|k}^j) \approx \mathcal{N}(x_k; x_{k|k}, P_{k|k})
\]

\[
\nu_k = \sum_{j=0}^{m_k} p_k^j \nu_k^j, \quad \nu_k^j = z_k^j - Hx_{k|k-1} \quad \text{combined innovation}
\]

\[
W_k = P_{k|k-1}H^T S_k^{-1}, \quad S_k = H P_{k|k-1}H^T + R_k \quad \text{Kalman gain matrix}
\]

\[
p_k^j = p_k^{j*} / \sum_j p_k^{j*}, \quad p_k^{j*} = \left\{ \frac{(1 - P_D) \rho F}{\sqrt{2\pi S_{n_k}}} e^{-\frac{1}{2} \nu_{n_k}^T S_{n_k} \nu_{n_k}} \right\} \quad \text{weighting factors}
\]

\[
x_k = x_{k|k-1} + W_k \nu_k \quad \text{(Filtering Update: Kalman)}
\]

\[
P_k = P_{k|k-1} - (1 - p_k^0) W_k S W_k^T \quad \text{(Kalman part)}
\]

\[
+ W_k \left\{ \sum_{j=0}^{m_k} p_k^j \nu_k^j \nu_k^j - \nu_k \nu_k^T \right\} W_k^T \quad \text{(Spread of Innovations)}
\]
PDAF: Characteristic Properties

- filtering: processing of *combined innovation*
- *all data* $Z_k$ in the gate are considered
- $p_i$ data dependent! Update *not linear*
- missing measurement: $P_{k|k-1}$ with weight $p_0$
- “usual” Kalman covarianve according to $(1 - p_0)$
- Spread *positively semidefinite*: larger covariance
- therefore: *data driven adaptivity*
- *non linear estimator*: data dependent error
- Performance prediction *only via simulations*

**Problem:** Multimodality is lost!