Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\ \mathbf{x}_{0|0},\ \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}_{k-1|k-1},\ \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\ \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\ \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k},\ \mathbf{P}_{k|k}\big)$

$$\begin{aligned}
\mathbf{x}_{k|k} &= \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, & \boldsymbol{\nu}_{k|k-1} &= \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}\\
\mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, & \mathbf{S}_{k|k-1} &= \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k\\
&\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} & &\text{'KALMAN gain matrix'}
\end{aligned}$$

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\ \mathbf{x}_{l|k},\ \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\ \mathbf{x}_{l+1|k},\ \mathbf{P}_{l+1|k}\big)$

$$\begin{aligned}
\mathbf{x}_{l|k} &= \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), & \mathbf{W}_{l|l+1} &= \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}\\
\mathbf{P}_{l|k} &= \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top
\end{aligned}$$

Fraunhofer FKIE

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

Fraunhofer FKIE

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

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where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big) \end{aligned}$$

Fraunhofer FKIE

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big) \end{aligned}$$

Looks like a Kalman filtering update!

Fraunhofer FKIE

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$ Looks like filtering!

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$ Looks like prediction!

Fraunhofer FKIE

$$
\begin{aligned}
p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) &\propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \qquad \text{Looks like filtering!} \\
&= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}_{l+\theta|l+1}^{\top} \\
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^{\top}\mathbf{P}_{l+1|l}^{-1} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^{\top} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}
$$

$$
p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \qquad \text{Looks like prediction!}
$$

Fraunhofer FKIE

$$
\begin{aligned}
p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) &\propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \qquad \text{Looks like filtering!} \\
&= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \boldsymbol{\Delta}_{l+\theta|l+1}\Big) \\
&= \mathcal{N}\Big(\mathbf{b}_{l+\theta|l+1};\, \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \boldsymbol{\Delta}_{l+\theta|l+1}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} \\
\boldsymbol{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\boldsymbol{\Phi}^{\top}_{l+\theta|l+1}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^{\top}_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^{\top}_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}
$$

$$
p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \qquad \text{Looks like prediction!}
$$

$$
\begin{aligned}
p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) &\propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \qquad \text{Looks like filtering!}\\
&= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \boldsymbol{\Delta}_{l+\theta|l+1}\Big)\\
&= \mathcal{N}\Big(\mathbf{b}_{l+\theta|l+1};\, \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \boldsymbol{\Delta}_{l+\theta|l+1}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l})\\
&= \mathbf{x}_{l+\theta|l} - \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1}\\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l}\\
\boldsymbol{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\boldsymbol{\Phi}_{l+\theta|l+1}^\top
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^\top\mathbf{P}_{l+1|l}^{-1}\\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^\top + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}
$$

$$
\begin{aligned}
p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \qquad \text{Looks like prediction!}\\
&= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{x}_{l+\theta|k},\, \mathbf{x}_{l+\theta|k}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\left(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}\right)\\
\mathbf{P}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\left(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l}\right)\boldsymbol{\Phi}_{l+\theta|l+1}^\top
\end{aligned}
$$

A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

$$p(\mathbf{z}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\, p(\mathbf{z}_k, \mathbf{x}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\, p(\mathbf{z}_k|\,\mathbf{x}_k)\, p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})$$

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$$= \int d\mathbf{x}_k\; \underbrace{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)}_{\text{likelihood: sensor model}}\; \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}_{\text{prediction at time } t_k}$$

Fraunhofer FKIE

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innovation statistics, expectation gates, gating

$$p(\mathbf{z}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\, p(\mathbf{z}_k, \mathbf{x}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\, p(\mathbf{z}_k|\,\mathbf{x}_k)\, p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})$$

$$= \int d\mathbf{x}_k\, \underbrace{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)}_{\text{likelihood: sensor model}}\, \underbrace{\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}_{\text{prediction at time } t_k}$$

$$= \mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_{k|k-1},\, \mathbf{S}_{k|k-1}) \qquad \text{(product formula)}$$

innovation: $\boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$,

innovation covariance: $\mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k$

expectation gate: $\boldsymbol{\nu}_{k|k-1}^\top \mathbf{S}_{k|k-1}^{-1} \boldsymbol{\nu}_{k|k-1} \leq \lambda(P_c)$

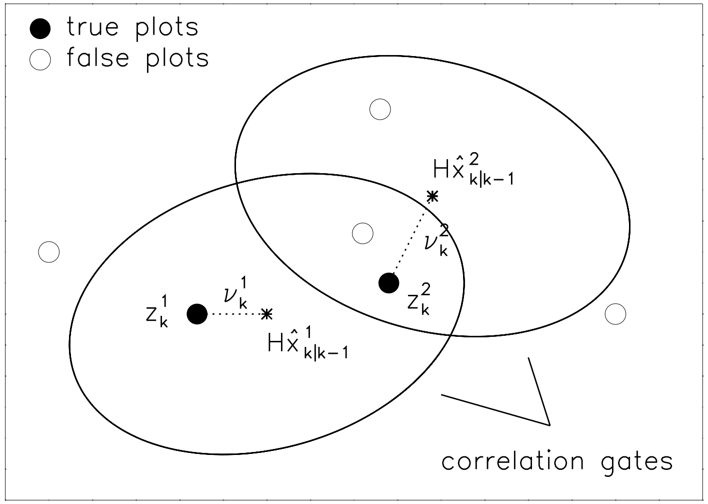
MAHALANOBIS ellipsoid containing $\mathbf{z}_k$ with certain probability P_c

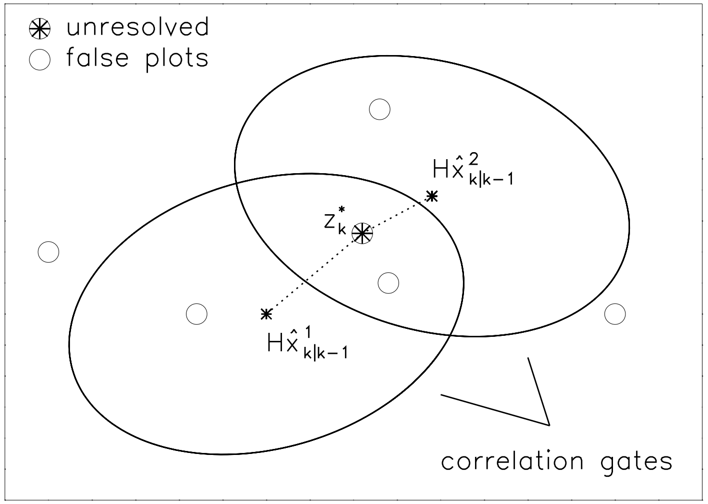
Choose $\lambda(P_c)$ (“gating parameter”) properly!

Can be looked up in a χ^2-table - discussed later!

Fraunhofer FKIE

Sensor data of uncertain origin



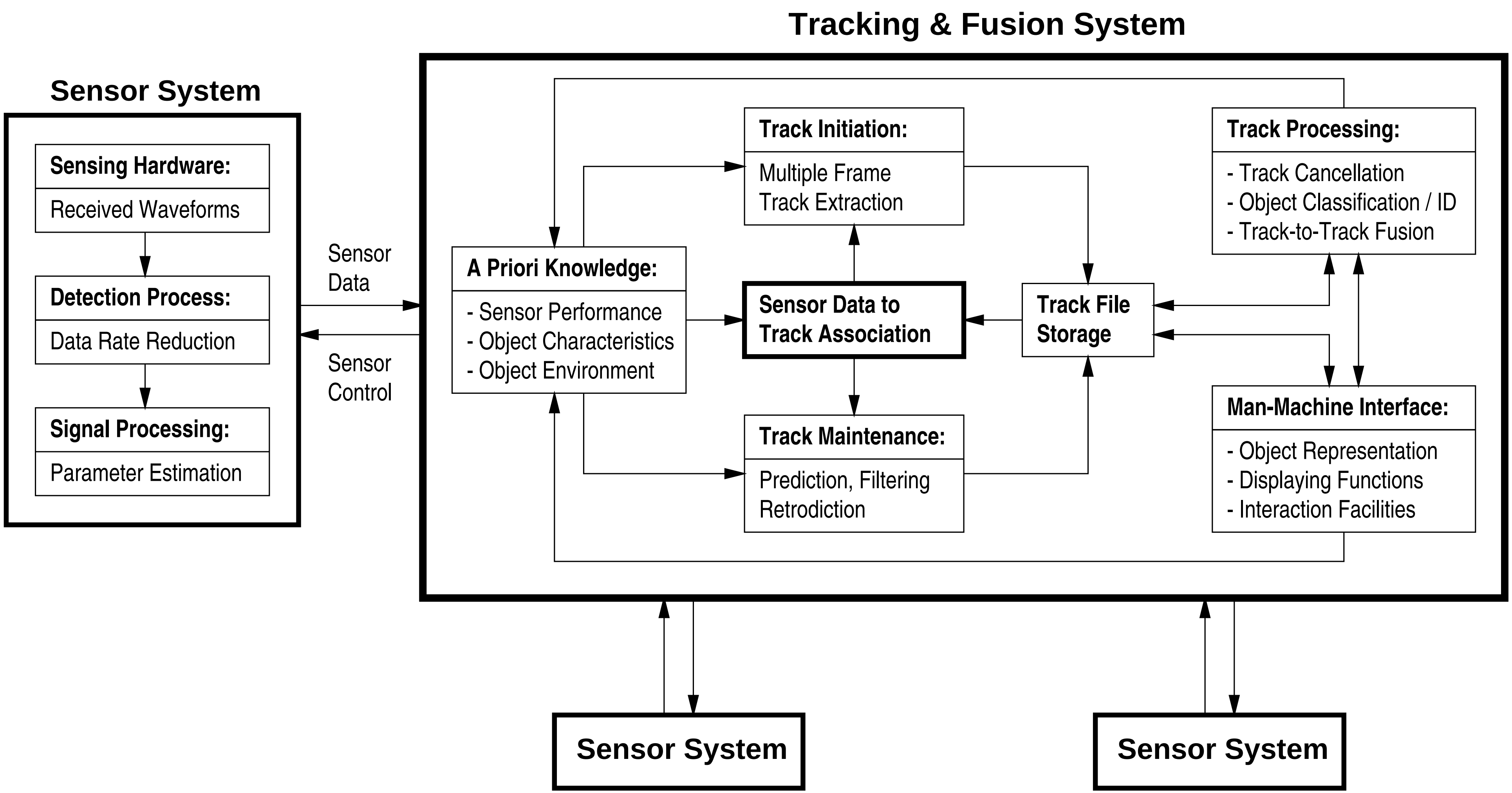


- prediction: $\mathbf{x}_{k|k-1}$, $\mathbf{P}_{k|k-1}$ (dynamics)
- innovation: $\boxed{\boldsymbol{\nu}_k = \mathbf{z}_k - \mathbf{H}\mathbf{x}_{k|k-1}}$, white
- Mahalanobis norm: $\boxed{||\boldsymbol{\nu}_k||^2 = \boldsymbol{\nu}_k^\top \mathbf{S}_k^{-1} \boldsymbol{\nu}_k}$
- expected plot: $\mathbf{z}_k \sim N(\mathbf{H}\mathbf{x}_{k|k-1}, \mathbf{S}_k)$
- $\boldsymbol{\nu}_k \sim N(0, \mathbf{S}_k)$, $\boxed{\mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}}$
- gating: $||\boldsymbol{\nu}_k|| < \lambda$, $P_c(\lambda)$ correlation prob.

missing/false plots, measurement errors, scan rate, agile targets: large gates

Fraunhofer FKIE

A Generic Tracking and Sensor Data Fusion System



Fraunhofer FKIE

Description of the Detection Process

Detector: receives signals and decides on object existence

Processor: processes detected signals and produces measurements

'D': detector detects an object

D: object actually existent

Fraunhofer FKIE

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Detector: receives signals and decides on object existence

Processor: processes detected signals and produces measurements

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D: object actually existent

error of 1. kind: $P_{\mathrm{I}} = P(\neg\text{'}D\text{'}|D)$

error of 2. kind: $P_{\mathrm{II}} = P(\text{'}D\text{'}|\neg D)$

measure of detection performance: $P_D = P(\text{'}D\text{'}|D)$

detector properties characterized by two parameters:

- detection probability $P_D = 1 - P_{\mathrm{I}}$
- false alarm probability $P_F = P_{\mathrm{II}}$

Fraunhofer FKIE

Description of the Detection Process

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detector properties characterized by two parameters:

- detection probability $P_D = 1 - P_{\mathrm{I}}$
- false alarm probability $P_F = P_{\mathrm{II}}$

example (Swerling I model): $P_D = P_D(P_F, \text{SNR}) = P_F^{1/(1+\text{SNR})}$

detector design: Maximize detection probability P_D
for a given, predefined false alarm probability P_F!

ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

$n_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$!

- E_0: the object was not detected; n_k false data in the Field of View (FoV)
- E_j, $j = 1, \ldots, n_k$: Object detected; $\mathbf{z}_k^j$ is object measurement; $n_k - 1$ false measurements

Consider the interpretations in the likelihood function $p(Z_k, n_k | \mathbf{x}_k)$!

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Consider the interpretations in the likelihood function $p(Z_k, n_k|\mathbf{x}_k)$!

$$p(Z_k, n_k|\mathbf{x}_k) = p(Z_k, n_k, \neg D|\mathbf{x}_k) + p(Z_k, n_k, D|\mathbf{x}_k) \qquad D = \text{“object was detected”}$$

Fraunhofer FKIE

ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

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$$= p(Z_k, n_k|\neg D, \mathbf{x}_k) \underbrace{P(\neg D|\mathbf{x}_k)}_{=1-P_D} + p(Z_k, n_k|D, \mathbf{x}_k) \underbrace{P(D|\mathbf{x}_k)}_{=P_D}$$

sensor parameter: detection probability P_D

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ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

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Consider the interpretations in the likelihood function $p(Z_k, n_k|\mathbf{x}_k)$!

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$$= p(Z_k, n_k|\neg D, \mathbf{x}_k)\, P(\neg D|\mathbf{x}_k) + p(Z_k, n_k|D, \mathbf{x}_k)\, p(D|\mathbf{x}_k)$$

$$= \underbrace{p(Z_k|n_k, \neg D, \mathbf{x}_k)}_{=|\text{FoV}|^{-n_k}}\ \underbrace{p(n_k|\neg D, \mathbf{x}_k)}_{=p_F(n_k)}\,(1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, \mathbf{x}_k)$$

false measurements: Poisson distributed in #, uniformly distributed in the FoV

Fraunhofer FKIE

Modeling of False Measurements (FM)

- **Probability of having n FM:** $p_F(n)$
  - mean number of FM in the 'Field of View' (FoV) of a sensor:

    $\bar{n} = \rho_F|\mathsf{FoV}|$, false measurement density ρ_F (perhaps not constant)

Modeling of False Measurements (FM)

- **Probability of having n FM:** $p_F(n)$
  - mean number of FM in the 'Field of View' (FoV) of a sensor:

    $\bar{n} = \rho_F|\mathrm{FoV}|$, false measurement density ρ_F (perhaps not constant)
  - assumption: n is a Poisson distributed RV with $\boxed{p_F(n) = \frac{\bar{n}^n}{n!}\, \mathrm{e}^{-\bar{n}}}$

Fraunhofer FKIE

Modeling of False Measurements (FM)

- **Probability of having n FM:** $p_F(n)$
  - mean number of FM in the 'Field of View' (FoV) of a sensor:

    $\bar{n} = \rho_F |\text{FoV}|$, false measurement density ρ_F (perhaps not constant)
  - assumption: n is a Poisson distributed RV with $\boxed{p_F(n) = \frac{\bar{n}^n}{n!}\,\mathrm{e}^{-\bar{n}}}$

    expectation: $\mathbb{E}[n] = \bar{n}$, variance: $\mathbb{V}[n] = \bar{n}$

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normalization: $\displaystyle\sum_{n=0}^{\infty} p_F(n) = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = \mathrm{e}^{-\bar{n}}\, \mathrm{e}^{\bar{n}}$ = 1

expectation: $\displaystyle\mathbb{E}[n] = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} n \frac{\bar{n}^n}{n!} = \mathrm{e}^{-\bar{n}} \sum_{n=1}^{\infty} n \frac{\bar{n}^n}{n!} = \bar{n}\mathrm{e}^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^{n-1}}{(n-1)!} =$ $\bar{n}$

variance: $\mathbb{V}[n] = \mathbb{E}[(n-\bar{n})^2] = \mathbb{E}[n^2] - \bar{n}^2 = \ldots \text{exercise!} \ldots = \bar{n}$

Modeling of False Measurements (FM)

- **Probability of having n FM:** $p_F(n)$
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- **Distribution of FM in the Field of View:** $p(\mathbf{z}_1^f, \ldots, \mathbf{z}_n^f|\mathbf{FoV})$
  - FM mutually independent: $p(\mathbf{z}_1^f, \ldots, \mathbf{z}_n^f|\mathrm{FoV}) = \prod_{i=1}^{n} p(\mathbf{z}_i^f|\mathrm{FoV})$

Fraunhofer FKIE

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  – uniformly distributed in the FoV: $p(\mathbf{z}_i^f|\mathsf{FoV}) = |\mathsf{FoV}|^{-1}$ (often!)

Fraunhofer FKIE

ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

$n_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$!

- E_0: the object was not detected; n_k false data in the Field of View (FoV)
- E_j, $j = 1, \dots, n_k$: Object detected; $\mathbf{z}_k^j$ is object measurement; $n_k - 1$ false measurements

Consider the interpretations in the likelihood function $p(Z_k, n_k|\mathbf{x}_k)$!

$$p(Z_k, n_k|\mathbf{x}_k) = p(Z_k, n_k, \neg D|\mathbf{x}_k) + p(Z_k, n_k, D|\mathbf{x}_k) \qquad D = \text{“object was detected”}$$

$$= p(Z_k, n_k|\neg D, \mathbf{x}_k)\, P(\neg D|\mathbf{x}_k) + p(Z_k, n_k|D, \mathbf{x}_k)\, p(D|\mathbf{x}_k)$$

$$= p(Z_k|n_k, \neg D, \mathbf{x}_k)\, p(n_k|\neg D, \mathbf{x}_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, \mathbf{x}_k)$$

$$= |\mathsf{FoV}|^{-n_k}\, p_F(n_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} \underbrace{p(Z_k|n_k, j, D, \mathbf{x}_k)}_{|\mathsf{FoV}|^{-(n_k-1)} N(\mathbf{z}_k^j; \mathbf{H}\mathbf{x}_k, \mathbf{R})}\ \underbrace{p(j|n_k, D)}_{=1/n_k}\ \underbrace{p(n_k|D)}_{=p_F(n_k-1)}$$

Insert Poisson distribution: $\quad p_F(n_k) = \frac{(\rho_F|\mathsf{FoV}|)^{-n_k}}{n_k!}\, \mathrm{e}^{-\rho_F|\mathsf{FoV}|}$

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$$= |\mathrm{FoV}|^{-n_k}\, p_F(n_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k|n_k, j, D, \mathbf{x}_k)\, p(j|n_k, D)\, p(n_k|D)$$

$$= \frac{\mathrm{e}^{-\rho_F|\mathrm{FoV}|}}{n_k!}\, \rho_F^{n_k-1}\Big((1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big)\Big)$$

Fraunhofer FKIE

Likelihood Functions

The likelihood function answers the question:

What does the sensor tell about the state x of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1$, $\rho_F = 0$

  at each time one measurement: $\boxed{p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k; \mathbf{H}\mathbf{x}_k, \mathbf{R})}$

- **real conditions, one object:** $P_D < 1$, $\rho_F > 0$

  at each time n_k measurements $Z_k = \{\mathbf{z}_k^1, \dots, \mathbf{z}_k^{n_k}\}$!

$$\boxed{p(Z_k, n_k|\mathbf{x}_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big)}$$

PDAF Filter: formally analogous to Kalman Filter

Filtering (scan $k-1$): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})$ (→ initiation)

prediction (scan k): $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$ (like Kalman)

Filtering (scan k): $p(\mathbf{x}_k|\mathcal{Z}^{k}) \approx \sum_{j=0}^{m_k} p_k^j\, \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}^j, \mathbf{P}_{k|k}^j)$ $\boxed{\approx \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}, \mathbf{P}_{k|k})}$

Fraunhofer FKIE

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$$\mathbf{x}_{k|k}^j = \begin{cases} \mathbf{x}_{k|k-1} & j=0 \\ \mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j & j\neq 0 \end{cases} \qquad \mathbf{P}_{k|k}^j = \begin{cases} \mathbf{P}_{k|k-1} & j=0 \\ \mathbf{P}_{k|k-1} - \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top & j\neq 0 \end{cases}$$

$$\boldsymbol{\nu}_k^j = \underbrace{\mathbf{z}_k^j - \mathbf{H}\mathbf{x}_k}_{\text{innovation}}, \qquad \mathbf{W}_k = \underbrace{\mathbf{P}_{k|k-1}\mathbf{H}^\top \mathbf{S}_k^{-1}}_{\text{gain matrix}}, \qquad \mathbf{S}_k = \underbrace{\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}_k}_{\text{innovation covariance}}$$

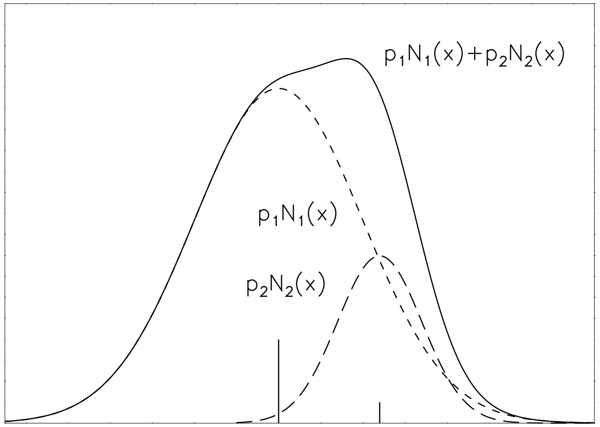
$$p_k^j = \underbrace{\frac{p_k^{j*}}{\sum_j p_k^{j*}}}_{\text{Gewichte}}, \qquad p_k^{j*} = \begin{cases} (1-P_D)\,\rho_F & j=0 \\ \frac{P_D}{\sqrt{|2\pi \mathbf{S}_{H_k}|}}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{\nu}_{H_k}^\top \mathbf{S}_{H_k} \boldsymbol{\nu}_{H_k}} & j\neq 0 \end{cases}$$

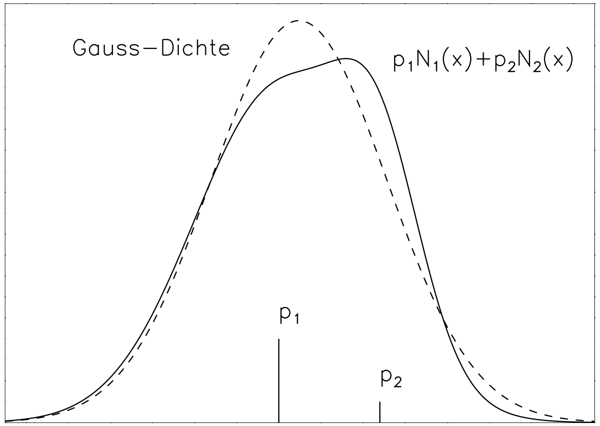
Fraunhofer FKIE

$$\textbf{\textit{Moment Matching:}}\ \text{Approximate an arbitrary pdf}$$

$p(x)$ with $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{C}[x] = \mathbf{P}$ by $p(x) \approx \mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{P}\big)$!

here especially: $p(x) = \sum_H p_H \, \mathcal{N}(x; \mathbf{x}_H, \mathbf{P}_H)$ (normal mixtures)





$$\mathbf{x} = \sum_H p_H \, \mathbf{x}_H$$

$$\mathbf{P} = \sum_H p_H \, \{\mathbf{P}_H + \overbrace{(\mathbf{x}_H - \mathbf{x})(\mathbf{x}_H - \mathbf{x})^\top}^{\text{spread term}}\}$$

Fraunhofer FKIE

Second-order Approximation of the Mixture Density:

$$\boxed{\sum_{j=1}^{m_k} p_k^j \, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k}^j,\, \mathbf{P}_{k|k}^j\big) \approx \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)}$$

mit: $$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \, \mathbf{x}_{k|k}^j$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \left(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\right)$$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j, \qquad \mathbf{x}_{k|k}^0 = \mathbf{x}_{k|k-1},\ \mathbf{x}_{k|k}^j = \mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)$$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

$$= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \big(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\big)$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)$$

$$
\begin{aligned}
\mathbf{x}_{k|k} &= \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j \\
&= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \big(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\big) \\
&= \mathbf{x}_{k|k-1} \underbrace{\left(p_k^0 + \sum_{j=1}^{m_k} p_k^j \right)}_{=1!} + \mathbf{W}_k \underbrace{\sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j}_{\text{mean!}}
\end{aligned}
$$

$$
\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)
$$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

$$= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \big(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\big) = \mathbf{x}_{k|k-1} + \mathbf{W}_k\, \boldsymbol{\nu}_k$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)$$

***Combined* Innovation:** $\boldsymbol{\nu}_k = \sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

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$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big), \qquad \mathbf{P}_{k|k}^0 = \mathbf{P}_{k|k-1},\ \mathbf{P}_{k|k}^j = \mathbf{P}_{k|k-1} - \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top$$

$$= \mathbf{P}_{k|k-1} - \sum_{j=1}^{m_k} p_k^j \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top + \sum_{j=1}^{m_k} p_k^j \mathbf{W}_k (\boldsymbol{\nu}_k^j - \boldsymbol{\nu}_k)(\boldsymbol{\nu}_k^j - \boldsymbol{\nu}_k)^\top \mathbf{W}_k^\top$$

***Combined* Innovation:** $\boldsymbol{\nu}_k = \sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

$$= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \big(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\big) = \mathbf{x}_{k|k-1} + \mathbf{W}_k\, \boldsymbol{\nu}_k$$

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$$= \mathbf{P}_{k|k-1} - \sum_{j=1}^{m_k} p_k^j \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top + \sum_{j=1}^{m_k} p_k^j \mathbf{W}_k (\boldsymbol{\nu}_k^j - \boldsymbol{\nu}_k)(\boldsymbol{\nu}_k^j - \boldsymbol{\nu}_k)^\top \mathbf{W}_k^\top$$

$$= \mathbf{P}_{k|k-1} - (1 - p_k^0)\mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top + \mathbf{W}_k \Big[\sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j \boldsymbol{\nu}_k^{j\top} - \boldsymbol{\nu}_k \boldsymbol{\nu}_k{}^\top\Big] \mathbf{W}_k^\top$$

***Combined* Innovation:** $\boldsymbol{\nu}_k = \sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j$

PDAF Filter: formally analog to Kalman Filter

Filtering (scan $k-1$): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})$ ($\rightarrow$ initiation)

prediction (scan k): $p(\mathbf{x}_{k}|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_{k};\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$ (like Kalman)

Filtering (scan k): $p(\mathbf{x}_{k}|\mathcal{Z}^{k}) \approx \sum_{j=0}^{m_k} p_k^j \mathcal{N}(\mathbf{x}_{k};\, \mathbf{x}_{k|k}^j, \mathbf{P}_{k|k}^j) \approx \mathcal{N}(\mathbf{x}_{k};\, \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

$$\boldsymbol{\nu}_k = \boxed{\textstyle\sum_{j=0}^{m_k} p_k^j \boldsymbol{\nu}_k^j}, \quad \boldsymbol{\nu}_k^j = \mathbf{z}_k^j - \mathbf{H}\mathbf{x}_{k|k-1}$$ *combined* innovation

$$\mathbf{W}_k = \mathbf{P}_{k|k-1}\mathbf{H}^\top\mathbf{S}_k^{-1}, \quad \mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}_k$$ Kalman gain matrix

$$p_k^j = p_k^{j*} \Big/ \textstyle\sum_j p_k^{j*}, \quad p_k^{j*} = \begin{cases} (1-P_D)\,\rho_F \\ \frac{P_D}{\sqrt{|2\pi\mathbf{S}_{H_k}|}}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{\nu}_{H_k}^\top \mathbf{S}_{H_k}\boldsymbol{\nu}_{H_k}} \end{cases}$$ weighting factors

$$\mathbf{x}_k = \mathbf{x}_{k|k-1} + \mathbf{W}_k\boldsymbol{\nu}_k$$ (Filtering Update: Kalman)

$$\mathbf{P}_k = \mathbf{P}_{k|k-1} - (1-p_k^0)\,\mathbf{W}_k\mathbf{S}\mathbf{W}_k^\top$$ (Kalman part)

$$+\, \mathbf{W}_k\Big\{\textstyle\sum_{j=0}^{m_k} p_k^j \boldsymbol{\nu}_k^j\boldsymbol{\nu}_k^{j\top} - \boldsymbol{\nu}_k\boldsymbol{\nu}_k^\top\Big\}\mathbf{W}_k^\top$$ (Spread of Innovations)

PDAF: Characteristic Properties

- filtering: processing of *combined innovation*
- *all data* Z_k in the gate are considered
- p_i data dependent! Update *not linear*
- missing measurement: $\mathbf{P}_{k|k-1}$ with weight p_0
- *"usual"* Kalman covarianve according to $(1 - p_0)$
- Spread *positively semidefinite:* larger covariance
- therefore: *data driven adaptivity*
- *non linear estimator:* data dependent error
- Performance prediction *only via simulations*

Problem: Multimodality is lost!