Multiple Sensor Target Tracking: Basic Idea

Iterative updating of conditional probability densities!

kinematic target state x_k at time t_k, **accumulated sensor data** $\mathcal{Z}^k$

a priori knowledge: target dynamics models, sensor model

- **prediction:** $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \xrightarrow{\text{dynamics model}} p(\mathbf{x}_k|\mathcal{Z}^{k-1})$
- **filtering:** $p(\mathrm{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor model}]{\text{sensor data } Z_k} p(\mathrm{x}_k|\mathcal{Z}^k)$
- **retrodiction:** $p(\mathrm{x}_{l-1}|\mathcal{Z}^k) \xleftarrow[\text{dynamics model}]{\text{filtering output}} p(\mathrm{x}_l|\mathcal{Z}^k)$

A first look at retrodiction today!

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Recapitulation: An Important Data Fusion Algorithm

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'KALMAN gain matrix'}$$

A deeper look into the dynamics and sensor models necessary!

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Remember your own ground truth generator!

Exercise 3.1

Consider a car moving on a mountain pass road modeled by:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} vt \\ a_y \sin(\frac{4\pi v}{a_x} t) \\ a_z \sin(\frac{\pi v}{a_x} t) \end{pmatrix}$$

$$v = 20 \frac{\text{km}}{\text{h}},\ a_x = 10\ \text{km},\ a_y = a_z = 1\ \text{km},\ t \in [0, a_x/v].$$

1. Plot the trajectory. Are the parameters reasonable? Try alternatives.
2. Calculate and plot the velocity and acceleration vectors:

$$\dot{\mathbf{r}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}, \quad \ddot{\mathbf{r}}(t) = -q \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix}.$$

3. Calculate for each instance of time t the tangential vectors in $\mathbf{r}(t)$:

$$\mathbf{t}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \dot{\mathbf{r}}(t).$$

4. Plot $|\dot{\mathbf{r}}(t)|$, $|\ddot{\mathbf{r}}(t)|$, and $\ddot{\mathbf{r}}(t)\mathbf{t}(t)$ over the time interval.
5. Discuss the temporal behaviour based on the trajectory $\mathbf{r}(t)$!

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Create your own sensor simulator!

Exercise 4.1

Simulate normally distributed radar measurements!

ΔT = 2 s, 2 radars at $\mathbf{r}_s^{1,2} = (x_s^{1,2}, y_s^{1,2}, z_s^{1,2})^\top$,

$x_s^{1,2}$ = 0, 100 km, $y_s^{1,2}$ = 100, 0 km, $z_s^{1,2}$ = 10 km.

State at time $t_k = k\Delta T$, $k \in \mathbb{Z}$: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$

1. Simulate range and azimuth measurements of the target position $\mathbf{r}_k$ with a random number generator normrnd$(0, 1)$ producing normally distributed zero-mean and unit-variance random numbers:

$$\mathbf{z}_k^p = \begin{pmatrix} z_k^r \\ z_k^\varphi \end{pmatrix} = \begin{pmatrix} \sqrt{(x_k - x_s)^2 + (y_k - y_s)^2 + (z_k - z_s)^2 - z_s^2} \\ \text{arctan}(\frac{y_k - y_s}{x_k - x_s}) \end{pmatrix} + \begin{pmatrix} \sigma_r \text{ normrnd}(0,1) \\ \sigma_\varphi \text{ normrnd}(0,1) \end{pmatrix}$$

   with σ_r = 10 m, σ_φ = 0.1° denoting the standard deviations in range and azimuth. Assume that the radars are not able to measure the elevation angle (see discussion on the whiteboard!).

2. Transform the measurements in x-y-Cartesian coordinates $z_k^r(\cos z_k^\varphi, \sin z_k^\varphi)^\top + \mathbf{r}_s$ and plot them over x-y projection of the true target trajectory! Play with sensor positions and measurement error standard deviations!

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Recapitulation: Piecewise Constant White Acceleration

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

For known $\mathbf{x}_{k-1}$ and without external influences we have with $\Delta T_k = t_k - t_{k-1}$:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},$$ **see blackboard!**

Assume during the interval ΔT_k a constant acceleration $\mathbf{a}_k$ causing the state evolution:

$$\begin{pmatrix} \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \Delta T_k\,\mathbf{I} \end{pmatrix} \mathbf{a}_k =: \mathbf{G}_k\mathbf{a}_k,$$ **linear transform!**

Let $\mathbf{a}_k$ be a Gaussian RV with pdf: $p(\mathbf{a}_k) = \mathcal{N}\big(\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2\mathbf{I}\big)$, we therefore have:

$$p(\mathbf{G}_k\mathbf{a}_k) = \mathcal{N}\big(\mathbf{G}_k\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2\mathbf{G}_k\mathbf{G}_k^\top\big).$$

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Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\Big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\Big)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\, \mathbf{I} & \frac{1}{2}\Delta T_k^3\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \Delta T_k^2\, \mathbf{I} \end{pmatrix}$$

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Recapitulation Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around truth position:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_k)$$

$$\mathbf{T} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **Cartesian error covariance (time dependent):**

$$\boxed{\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\, \mathbf{R}\, \mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top}$$

- **sensor fusion: sensor-to-target-geometry enters into** $\mathbf{T}\mathbf{R}\mathbf{T}^\top$

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initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
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filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^{\top}, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^{\top} + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^{\top}\mathbf{S}_{k|k-1}{}^{-1}$$
'KALMAN gain matrix'

Exercise 4.2

In your sensor simulator, chose a sensor at position $\mathbf{r}_s$ that produces x-y measurements $\mathbf{z}_k$ of the Cartesian target x-y positions $\mathbf{H}\mathbf{x}_k$ from your ground truth generator using the measurement matrix $\mathbf{H}$:

$$\mathbf{H}\mathbf{x}_k = \begin{pmatrix} 1,0,0,0,0,0 \\ 0,1,0,0,0,0 \end{pmatrix} \mathbf{x}_k$$

Calculate for each measurement the measurement error covariance matrix $\mathbf{R}_k$ based on the true target position. Program your first Kalman filter initiated by the first measurement and reasonably chosen covariance matrices $\mathbf{P}_{1|1}$. What is reasonable? Visualize nicely and compare with the truth and the measurements.

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Recapitulation: Create a singe *effective measurement* by preprocessing of the individual measurements!

$$\mathbf{z}_k = \mathbf{R}_k \sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1} \mathbf{z}_k^s$$

weighted arithmetic mean of measurements

$$\mathbf{R}_k = \left(\sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1}\right)^{-1}$$

harmonic mean of measuremt covariances

A typical structure for fusion equations!

Exercise 4.3

With measurement specific measurement error covariances, your Kalman filter already is a multiple sensor fusion algorithms. Use two radar sensors, fuse the measurements, feed them into the Kalman filter and discuss the result!

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More general: measurement process

- **linear measurement equation:**

  $$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{u}_k, \quad p(\mathbf{u}_k) = \mathcal{N}\Big(\mathbf{u}_k;\ \mathbf{o},\ \mathbf{R}_k\Big)$$

  – to be measured: *linear* functions of the object state

  – measurement error: biasfree, Gaussian distrib.
  independent for different t_k

  – $\mathbf{y}_k = \mathbf{z}_k - \mathbf{H}_k \mathbf{x}_k$ has the pdf: $p(\mathbf{y}_k) = p(\mathbf{u}_k)$

- **Approach for the requested pdf ('likelihood fkt.):**

  $$\boxed{p\Big(\mathbf{z}_k \big| \, \mathbf{x}_k\Big) = \mathcal{N}\Big(\mathbf{z}_k;\ \mathbf{H}_k \mathbf{x}_k,\ \mathbf{R}_k\Big)}$$

- **Example: position measurement**

  $$\mathbf{H}_k = (\mathbf{I}, \mathbf{O}, \mathbf{O}), \quad \mathbf{H}_k \mathbf{x}_k = \mathbf{r}_k$$

  $\mathbf{R}_k$: measurement error covariance matrix

  possibly depending on the sensor-to-target geometry

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Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big)\ \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

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Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big)\ \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

$$p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big) = \frac{p(Z_k, \ldots, Z_{l+1}|\mathbf{x}_{l+1}, \mathbf{x}_l, \mathcal{Z}^l)\ p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l\big)}{\int d\mathbf{x}_l\ p(Z_k, \ldots, Z_{l+1}|\mathbf{x}_{l+1}, \mathbf{x}_l, \mathcal{Z}^l)\ p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l\big)} = p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l\big)$$

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Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big)\; \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

$$p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big) = p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l\big) = \frac{p\big(\mathbf{x}_{l+1}|\,\mathbf{x}_l\big)\; p\big(\mathbf{x}_l|\,\mathcal{Z}^l\big)}{\int d\mathbf{x}_l\; \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathbf{x}_l\big)}_{\text{dynamics model}}\; \underbrace{p\big(\mathbf{x}_l|\,\mathcal{Z}^l\big)}_{\text{filtering } t_l}}$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p(\mathbf{x}_l|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, \frac{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)\; p(\mathbf{x}_l|\,\mathcal{Z}^l)}{\int d\mathbf{x}_l\;\underbrace{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)}_{\text{dynamics model}}\;\underbrace{p(\mathbf{x}_l|\,\mathcal{Z}^l)}_{\text{filtering } t_l}}\;\underbrace{p(\mathbf{x}_{l+1}|\,\mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

- $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ retrodiction: last iteration step
- $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ dynamic object behavior
- $p(\mathbf{x}_l|\mathcal{Z}^l)$ filtering at the time considered
- GAUSSians, GAUSSian mixtures: Exploit product formula!
- linear GAUSSian likelihood/dynamics: Rauch-Tung-Striebel smoothing

Exercise 4.4 Derive the *Rauch-Tung-Striebel* formulae

by using the Kalman filter assumptions

and the product formula (twice)!

retrodiction: $\mathcal{N}(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k})$

$$\begin{aligned}\mathbf{x}_{l|k} &= \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^{\top}\mathbf{P}_{l+1|l}^{-1}\\ \mathbf{P}_{l|k} &= \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^{\top}\end{aligned}$$

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Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\ \mathbf{x}_{0|0},\ \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}_{k-1|k-1},\ \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\ \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\ \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k},\ \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'KALMAN gain matrix'}$$

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\ \mathbf{x}_{l|k},\ \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\ \mathbf{x}_{l+1|k},\ \mathbf{P}_{l+1|k}\big)$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$

Exercise 4.5 Implement the *Rauch-Tung-Striebel* formulae in your simulator (in the course of the semester ...)!

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Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

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Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

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where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big) \end{aligned}$$

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Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\left(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\right) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\left(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\right) \end{aligned}$$

Looks like a Kalman filtering update!

Fraunhofer FKIE

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

Fraunhofer FKIE

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1}
\end{aligned}$$

$$\mathbf{\Delta}_{l+\theta|l+1} = \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}_{l+\theta|l+1}^\top$$

$$\begin{aligned}
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^\top\mathbf{P}_{l+1|l}^{-1} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^\top + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

Fraunhofer FKIE

$$
\begin{aligned}
p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) &\propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\
&= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big) \\
&= \mathcal{N}\Big(\mathbf{b}_{l+\theta|l+1};\, \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big)
\end{aligned}
$$

Looks like filtering!

$$
\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} \\
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}^{\top}_{l+\theta|l+1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^{\top}_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^{\top}_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}
$$

$$
p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)
$$

Looks like prediction!

Fraunhofer FKIE

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)$$
$$= \mathcal{N}\big(\mathbf{b}_{l+\theta|l+1};\, \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} \\
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}^{\top}_{l+\theta|l+1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^{\top}_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^{\top}_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{x}_{l+\theta|k},\, \mathbf{x}_{l+\theta|k}\big)$$

$$\begin{aligned}
\mathbf{x}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\left(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}\right) \\
\mathbf{P}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\left(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l}\right)\mathbf{\Phi}^{\top}_{l+\theta|l+1}
\end{aligned}$$

Fraunhofer
FKIE

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$