Multiple Sensor Target Tracking: Basic Idea

Iterative updating of conditional probability densities!

kinematic target state x_k at time t_k, **accumulated sensor data** $\mathcal{Z}^k$

a priori knowledge: target dynamics models, sensor model

- **prediction:** $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \xrightarrow{\text{dynamics model}} p(\mathbf{x}_k|\mathcal{Z}^{k-1})$
- **filtering:** $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor model}]{\text{sensor data } Z_k} p(\mathbf{x}_k|\mathcal{Z}^k)$
- **retrodiction:** $p(\mathbf{x}_{l-1}|\mathcal{Z}^k) \xleftarrow[\text{dynamics model}]{\text{filtering output}} p(\mathbf{x}_l|\mathcal{Z}^k)$

Fraunhofer FKIE

The Multivariate GAUSSian Pdf

- *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$

- *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

  $q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

- *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\, \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top\mathbf{P}^{-1}(x-\mathbf{x})}$$

- GAUSS*ian Mixtures:* $p(x) = \sum_i p_i\, \mathcal{N}(x;\, \mathbf{x}_i,\, \mathbf{P}_i)$ (weighted sums)

Very First Look at an Important Data Fusion Algorithm

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$ 'KALMAN gain matrix'

A deeper look into the dynamics and sensor models necessary!

Remember your own ground truth generator!

Exercise 3.1

Consider a car moving on a mountain pass road modeled by:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} vt \\ a_y \sin(\frac{4\pi v}{a_x} t) \\ a_z \sin(\frac{\pi v}{a_x} t) \end{pmatrix}$$

$v = 20 \frac{\text{km}}{\text{h}}$, a_x = 10 km, $a_y = a_z = 1$ km, $t \in [0, a_x/v]$.

1. Plot the trajectory. Are the parameters reasonable? Try alternatives.
2. Calculate and plot the velocity and acceleration vectors:

$$\dot{\mathbf{r}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}, \quad \ddot{\mathbf{r}}(t) = -q \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix}.$$

3. Calculate for each instance of time t the tangential vectors in $\mathbf{r}(t)$:

$$\mathbf{t}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \dot{\mathbf{r}}(t).$$

4. Plot $|\dot{\mathbf{r}}(t)|$, $|\ddot{\mathbf{r}}(t)|$, and $\ddot{\mathbf{r}}(t)\mathbf{t}(t)$ over the time interval.
5. Discuss the temporal behaviour based on the trajectory $\mathbf{r}(t)$!

How to deal with probability density functions?

- pdf $p(x)$: Extract *probability statements* about the RV x by integration!
- naïvely: *positive* and *normalized* functions ($p(x) \geq 0$, $\int dx\, p(x) = 1$)
- *conditional pdf* $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on y on RV x?
- *marginal density* $p(x) = \int dy\, p(x,y) = \int dy\, p(x|y)\, p(y)$: Enter y!
- Bayes: $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx\, p(y|x)p(x)}$: $p(x|y) \leftarrow p(y|x)$, $p(x)$!

Fraunhofer FKIE

How to deal with probability density functions?

- pdf $p(x)$: Extract *probability statements* about the RV x by integration!

- naïvely: *positive* and *normalized* functions ($p(x) \geq 0$, $\int dx\, p(x) = 1$)

- *conditional pdf* $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on y on RV x?

- *marginal density* $p(x) = \int dy\, p(x, y) = \int dy\, p(x|y)\, p(y)$: Enter y!

- Bayes: $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx\, p(y|x)p(x)}$: $p(x|y) \leftarrow p(y|x)$, $p(x)$!

- *certain knowledge* on x: $p(x) = \delta(x - y)$ '=' $\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-y)^2}{\sigma^2}}$

How to deal with probability density functions?

- pdf $p(x)$: Extract *probability statements* about the RV x by integration!
- naïvely: *positive* and *normalized* functions ($p(x) \geq 0$, $\int dx\, p(x) = 1$)
- *conditional pdf* $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on y on RV x?
- *marginal density* $p(x) = \int dy\, p(x,y) = \int dy\, p(x|y)\, p(y)$: Enter y!
- Bayes: $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx\, p(y|x)p(x)}$: $p(x|y) \leftarrow p(y|x)$, $p(x)$!
- *certain knowledge* on x: $p(x) = \delta(x-y)$ '=' $\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-y)^2}{\sigma^2}}$
- *transformed* $RV\, y = t[x]$: $p(y) = \int dx\, p(y,x) = \int dx\, p(y|x)\, p_x(x)$

Fraunhofer FKIE

How to deal with probability density functions?

- pdf $p(x)$: Extract *probability statements* about the RV x by integration!
- naïvely: *positive* and *normalized* functions ($p(x) \geq 0$, $\int dx\, p(x) = 1$)
- *conditional pdf* $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on y on RV x?
- *marginal density* $p(x) = \int dy\, p(x,y) = \int dy\, p(x|y)\, p(y)$: Enter y!
- Bayes: $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx\, p(y|x)p(x)}$: $p(x|y) \leftarrow p(y|x), p(x)$!
- *certain knowledge* on x: $p(x) = \delta(x-y)$ '=' $\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-y)^2}{\sigma^2}}$
- *transformed RV* $y = t[x]$: $p(y) = \int dx\, p(y,x) = \int dx\, p(y|x)\, p_x(x) =$
  $\int dx\, \delta(y - t[x])\, p_x(x) =: [\mathcal{T} p_x](y)$ ($\mathcal{T} : p_x \mapsto p$, "transfer operator")

Fraunhofer FKIE

The Multivariate GAUSSian Pdf

- *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$

- *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

  $q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

- *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\, \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top\mathbf{P}^{-1}(x-\mathbf{x})}$$

Exercise 3.2 Show: $\int dx\, \mathrm{e}^{-q(x)} = \sqrt{|2\pi\mathbf{P}|}$, $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P}$

Trick: Symmetric, positively definite matrices can be diagonalized by an orthogonal coordinate transform.

The Multivariate GAUSSian Pdf

- *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$

- *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

  $q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

- *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\ \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\ \mathbf{x},\ \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\ \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top\mathbf{P}^{-1}(x-\mathbf{x})}$$

$$\mathbb{E}[x] = \mathbf{x},\ \ \mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P} \quad \text{(covariance)}$$

The Multivariate Gaussian Pdf

- *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$

- *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

  $q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

- *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\, \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top \mathbf{P}^{-1}(x-\mathbf{x})}$$

$$\mathbb{E}[x] = \mathbf{x},\ \ \mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P} \ \ \text{(covariance)}$$

- *Covariance Matrix:* Expected error of the expectation.

A Useful Product Formula for GAUSSians

$$\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\,\mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{y} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Sketch of the proof (done in an exercise later!):

- Interpret $\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\,\mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P})$ as a joint pdf $p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{z},\mathbf{x})$.
- Show that $p(\mathbf{z},\mathbf{x})$ is a GAUSSian: $p(\mathbf{z},\mathbf{x}) = \mathcal{N}\left(\binom{\mathbf{z}}{\mathbf{x}};\, \binom{\mathbf{H}\mathbf{y}}{\mathbf{y}},\, \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}\right)$.
- Calculate from $p(\mathbf{z},\mathbf{x})$ the marginal and conditional pdfs $p(\mathbf{z})$ and $p(\mathbf{x}|\mathbf{z})$.
- From $p(\mathbf{z},\mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) = p(\mathbf{x},\mathbf{z})$ we obtain the result.

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

Affine Transforms of GAUSSian Random Variables

$$\boxed{\mathcal{N}\Big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\Big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\Big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\Big)}$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\Big(\mathbf{x};\ \mathbf{y},\ \mathbf{D}\Big)$ with $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D} \to 0$$

Affine Transforms of GAUSSian Random Variables

$$\boxed{\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)}$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{D}\big)$ with $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y}; \mathbf{t}+\mathbf{T}\mathbf{x}, \mathbf{D})\, \mathcal{N}(\mathbf{x}; \mathbb{E}[\mathbf{x}], \mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D} \to 0$$

$$= \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top + \mathbf{D}\big) \qquad \text{for } \mathbf{D} \to 0; \quad \text{product formula!}$$

Also true if $\mathrm{dim}(\mathbf{x}) \neq \mathrm{dim}(\mathbf{y})$**!**

Fraunhofer FKIE

A popular model for object evolutions

Piecewise Constant White Acceleration Model

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

For known $\mathbf{x}_{k-1}$ and without external influences we have with $\Delta T_k = t_k - t_{k-1}$:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{I} & \Delta T_k \, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1} \mathbf{x}_{k-1}, \qquad \textbf{see blackboard!}$$

Assume during the interval ΔT_k a constant acceleration $\mathbf{a}_k$ causing the state evolution:

$$\begin{pmatrix} \frac{1}{2} \Delta T_k^2 \, \mathbf{I} \\ \Delta T_k \, \mathbf{I} \end{pmatrix} \mathbf{a}_k =: \mathbf{G}_k \mathbf{a}_k, \qquad \textbf{linear transform!}$$

Let $\mathbf{a}_k$ be a Gaussian RV with pdf: $p(\mathbf{a}_k) = \mathcal{N}\big(\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2 \mathbf{I}\big)$, we therefore have:

$$p(\mathbf{G}_k \mathbf{a}_k) = \mathcal{N}\big(\mathbf{G}_k \mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2 \mathbf{G}_k \mathbf{G}_k^\top\big).$$

Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\Big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\Big)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T_k^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

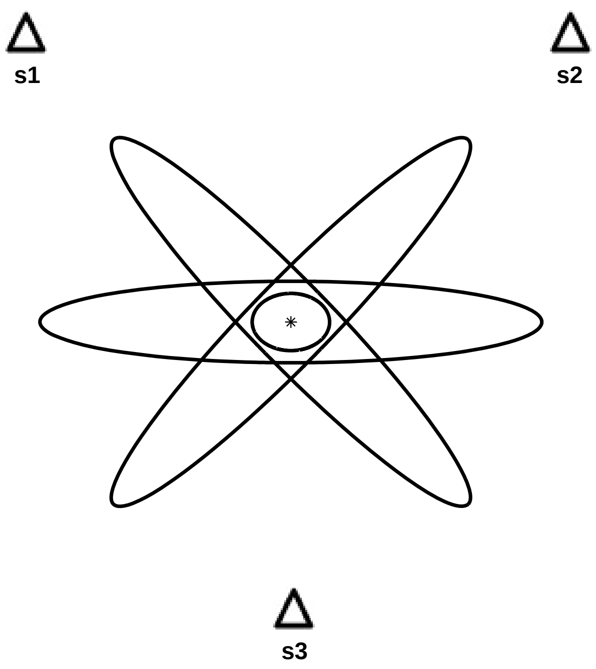
Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

a closer look: The error of each measurement $\mathbf{z}_i$ is described by a related measurement error *covariance matrix* $\mathbf{R}_i$ ('error ellipsoids'). In 2 dimensions:



$\mathbf{R}_i$ can strongly depend on the underlying senor-to-target geometry!

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

  $\mathbf{z}_k = (r_k, \varphi_k)^\top$, measurement error: $\mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}$, r, φ independent

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

  $\mathbf{z}_k = (r_k, \varphi_k)^\top$, measurement error: $\mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}$, r, φ independent

  Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

- **What is the likelihood function in Cartesian coordinates?**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix}$$

Fraunhofer FKIE

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

constant and linear term of a Taylor series only, blackboard!

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **affine transform of GAUSSian random variables:**

$$\mathcal{N}\big(\mathbf{z};\, \mathbf{x},\, \mathbf{R}\big) \xrightarrow{\mathbf{z}'=\mathbf{t}+\mathbf{T}\mathbf{z}} \mathcal{N}\big(\mathbf{z}';\, \mathbf{t}+\mathbf{T}\mathbf{x},\, \mathbf{T}\mathbf{R}\mathbf{T}^\top\big)$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

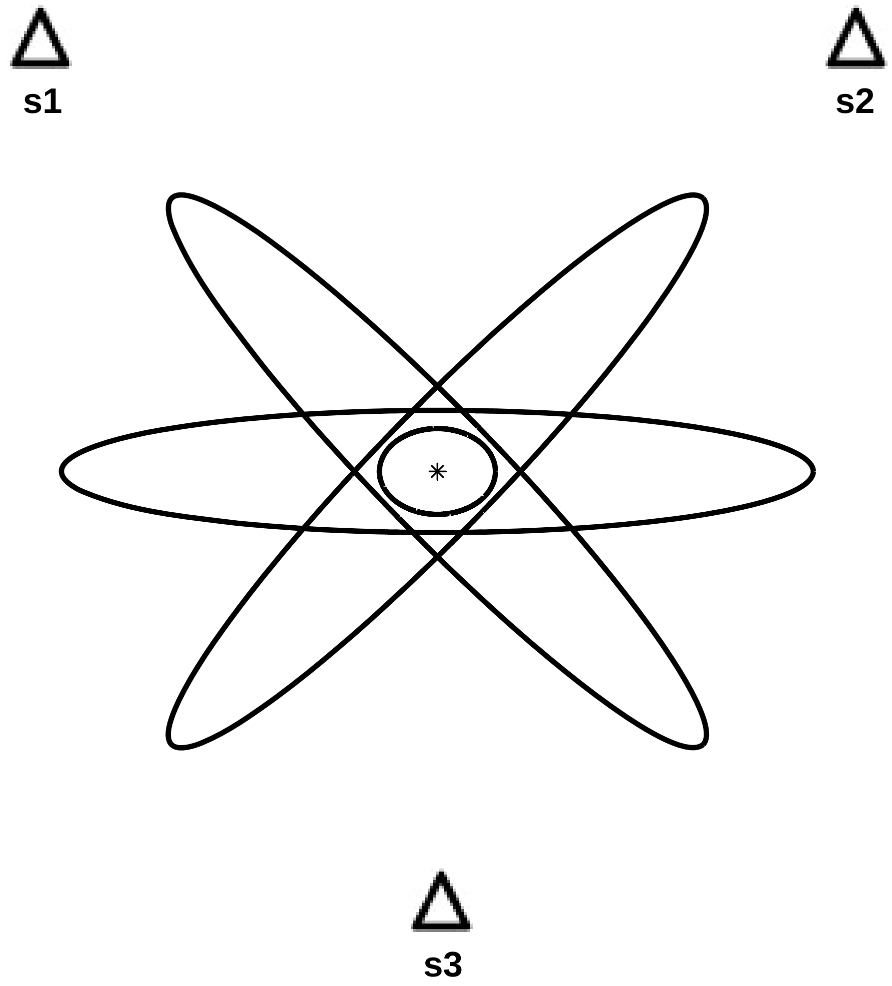
$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **Cartesian error covariance (time dependent):**

$$\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\, \mathbf{R}\, \mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top$$

- **sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$**



sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$

Fraunhofer FKIE

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Fraunhofer FKIE

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) = p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \qquad \text{independent sensors}$$

$$= \mathcal{N}\left(\mathbf{z}_k^1;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^1\right) \mathcal{N}\left(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\right)$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) = p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \qquad \text{independent sensors}$$

$$= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big)$$

A Useful Product Formula for GAUSSians

$$\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\,\mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{y} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$\begin{aligned} p(\mathbf{z}_k^1, \mathbf{z}_k^2|\mathbf{x}_k) &= p(\mathbf{z}_k^1|\mathbf{x}_k)\, p(\mathbf{z}_k^2|\mathbf{x}_k) \qquad \text{independent sensors} \\ &= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \\ &= \mathcal{N}\big(\mathbf{H}_k\mathbf{x}_k;\, \mathbf{z}_k^1,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \\ &\propto \mathcal{N}\Big(\mathbf{H}_k\mathbf{x}_k;\, \underbrace{\mathbf{R}_k(\mathbf{R}_k^{1\,-1}\mathbf{z}_k^1 + \mathbf{R}_k^{2\,-1}\mathbf{z}_k^2)}_{=\mathbf{z}_k},\, \underbrace{(\mathbf{R}_k^{1\,-1} + \mathbf{R}_k^{2\,-1})^{-1}}_{=\mathbf{R}_k}\Big) \\ &= \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big) \end{aligned}$$

Exercise 3.3 Generalize to the case $S_k > 2$ (induction argument)!

One possible fusion strategy: Create a singe *effective measurement* by preprocessing of the individual measurements!

$$\mathbf{z}_k = \mathbf{R}_k \sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1} \mathbf{z}_k^s$$ weighted arithmetic mean of measurements

$$\mathbf{R}_k = \left(\sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1}\right)^{-1}$$ harmonic mean of measuremt covariances

A typical structure for fusion equations!