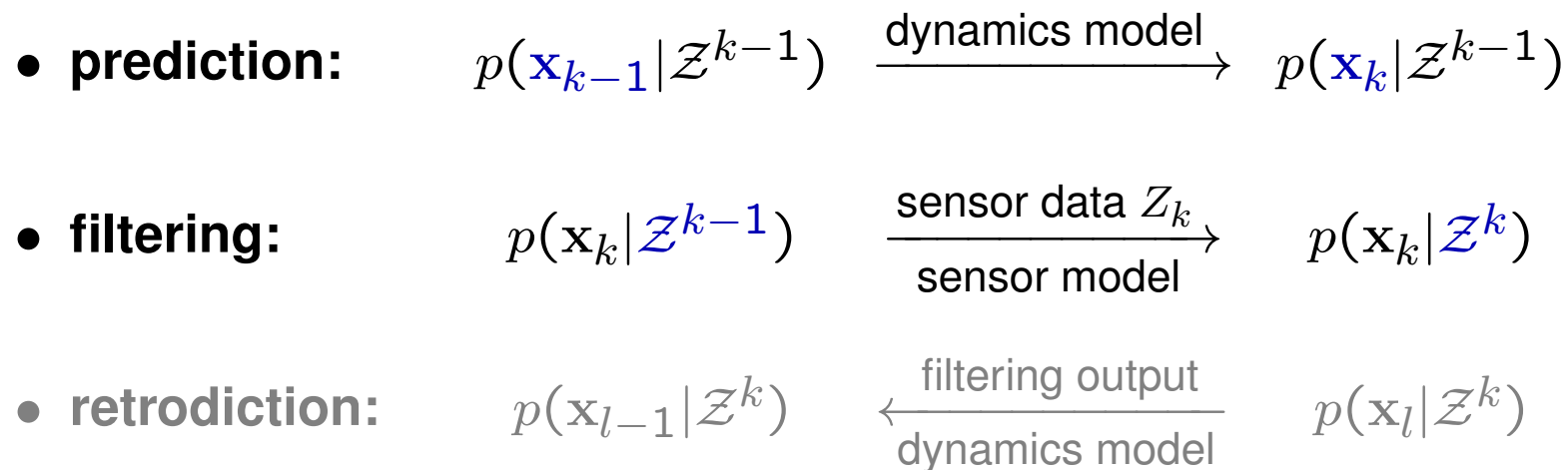


# Multiple Sensor Target Tracking: Basic Idea

*Iterative updating of conditional probability densities!*

**kinematic target state**  $\mathbf{x}_k$  at time  $t_k$ , **accumulated sensor data**  $\mathcal{Z}^k$

**a priori knowledge:** target dynamics models, sensor model



# The Multivariate GAUSSian Pdf

– *wanted:* probabilities ‘concentrated’ around a center  $\mathbf{x}$

– *quadratic distance:*  $q(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{x})\mathbf{P}^{-1}(\mathbf{x} - \mathbf{x})^\top$

$q(\mathbf{x})$  defines an ellipsoid around  $\mathbf{x}$ , its volume and orientation being determined by a matrix  $\mathbf{P}$  (symmetric:  $\mathbf{P}^\top = \mathbf{P}$ , positively definite: all eigenvalues  $> 0$ ).

– *first attempt:*  $p(\mathbf{x}) = e^{-q(\mathbf{x})} / \int d\mathbf{x} e^{-q(\mathbf{x})}$  (normalized!)

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{x}, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{x})^\top \mathbf{P}^{-1}(\mathbf{x}-\mathbf{x})}$$

– *GAUSSian Mixtures:*  $p(\mathbf{x}) = \sum_i p_i \mathcal{N}(\mathbf{x}; \mathbf{x}_i, \mathbf{P}_i)$  (weighted sums)

# Very First Look at an Important Data Fusion Algorithm

**Kalman filter:**  $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ ,  $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

**initiation:**  $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0; \mathbf{x}_{0|0}, \mathbf{P}_{0|0})$ , initial ignorance:  $\mathbf{P}_{0|0}$  'large'

**prediction:**  $\mathcal{N}(\mathbf{x}_{k-1}; \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1}, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1} \mathbf{x}_{k-1|k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1} \mathbf{P}_{k-1|k-1} \mathbf{F}_{k|k-1}^\top + \mathbf{D}_{k|k-1}$$

**filtering:**  $\mathcal{N}(\mathbf{x}_k; \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

$$\begin{aligned} \mathbf{x}_{k|k} &= \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1} \boldsymbol{\nu}_{k|k-1}, & \boldsymbol{\nu}_{k|k-1} &= \mathbf{z}_k - \mathbf{H}_k \mathbf{x}_{k|k-1} \\ \mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1} \mathbf{S}_{k|k-1} \mathbf{W}_{k|k-1}^\top, & \mathbf{S}_{k|k-1} &= \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^\top + \mathbf{R}_k \\ \mathbf{W}_{k|k-1} &= \mathbf{P}_{k|k-1} \mathbf{H}_k^\top \mathbf{S}_{k|k-1}^{-1} & & \text{'KALMAN gain matrix'} \end{aligned}$$

**A deeper look into the dynamics and sensor models necessary!**

# Create your own ground truth generator!

Consider a car moving on a mountain pass road modeled by:

## Exercise 3.1

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = A \begin{pmatrix} vt \\ a_y \sin\left(\frac{4\pi v}{a_x} t\right) \\ a_z \sin\left(\frac{\pi v}{a_x} t\right) \end{pmatrix}$$

$$v = 20 \frac{\text{km}}{\text{h}}, a_y = a_z = 1 \text{ km}, t \in [0, a_x/v_x].$$

1. Plot the trajectory. Are the parameters reasonable? Try alternatives.
2. Calculate and plot the velocity and acceleration vectors:

$$\dot{\mathbf{r}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}, \quad \ddot{\mathbf{r}}(t) = -q \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix}.$$

3. Calculate for each instance of time  $t$  the tangential vectors in  $\mathbf{r}(t)$ :

$$\mathbf{t}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \dot{\mathbf{r}}(t).$$

4. Plot  $|\dot{\mathbf{r}}(t)|$ ,  $|\ddot{\mathbf{r}}(t)|$ , and  $\ddot{\mathbf{r}}(t)\mathbf{t}(t)$  over the time interval.
5. Discuss the temporal behaviour based on the trajectory  $\mathbf{r}(t)$ !

# How to deal with probability density functions?

- pdf  $p(x)$ : Extract *probability statements* about the RV  $x$  by integration!
- naïvely: *positive* and *normalized* functions ( $p(x) \geq 0$ ,  $\int dx p(x) = 1$ )
- *conditional pdf*  $p(x|y) = \frac{p(x,y)}{p(y)}$ : Impact of information on  $y$  on RV  $x$ ?
- *marginal density*  $p(x) = \int dy p(x, y) = \int dy p(x|y) p(y)$ : Enter  $y$ !
- Bayes:  $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx p(y|x)p(x)}$ :  $p(x|y) \leftarrow p(y|x), p(x)$ !

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- *certain knowledge* on  $x$ :  $p(x) = \delta(x - y)$  '= $' \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} \frac{(x-y)^2}{\sigma^2}}$

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- *transformed RV*  $y = t[x]$ :  $p(y) = \int dx p(y, x) = \int dx p(y|x) p_x(x) = \int dx \delta(y - t[x]) p_x(x) =: [\mathcal{T} p_x](y)$  ( $\mathcal{T} : p_x \mapsto p$ , “transfer operator”)



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**Exercise 3.2** Show:  $\int d\mathbf{x} e^{-q(\mathbf{x})} = \sqrt{|2\pi\mathbf{P}|}$ ,  $\mathbb{E}[\mathbf{x}] = \mathbf{x}$ ,  $\mathbb{E}[(\mathbf{x} - \mathbf{x})(\mathbf{x} - \mathbf{x})^\top] = \mathbf{P}$

Trick: Symmetric, positively definite matrices can be diagonalized by an orthogonal coordinate transform.

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$$\mathbb{E}[\mathbf{x}] = \mathbf{x}, \quad \mathbb{E}[(\mathbf{x} - \mathbf{x})(\mathbf{x} - \mathbf{x})^\top] = \mathbf{P} \quad (\text{covariance})$$

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– *Covariance Matrix:* Expected error of the expectation.

# A Useful Product Formula for GAUSSIANS

$$\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{y}, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x}; \mathbf{y} + \mathbf{W}\boldsymbol{\nu}, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x}; \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}), \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

*Sketch of the proof (done in an exercise later!):*

- Interpret  $\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P})$  as a joint pdf  $p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{z}, \mathbf{x})$ .
- Show that  $p(\mathbf{z}, \mathbf{x})$  is a GAUSSIAN:  $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\left(\begin{pmatrix} \mathbf{z} \\ \mathbf{x} \end{pmatrix}; \begin{pmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}\right)$ .
- Calculate from  $p(\mathbf{z}, \mathbf{x})$  the marginal and conditional pdfs  $p(\mathbf{z})$  and  $p(\mathbf{x}|\mathbf{z})$ .
- From  $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) = p(\mathbf{x}, \mathbf{z})$  we obtain the result.

# Affine Transforms of GAUSSIAN Random Variables

$$\mathcal{N}(\mathbf{x}; \mathbb{E}[\mathbf{x}], \mathbb{C}[\mathbf{x}]) \xrightarrow{y=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}(\mathbf{y}; \mathbf{t} + \mathbf{T}\mathbb{E}[\mathbf{x}], \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top)$$

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A possible representation:  $\delta(\mathbf{x} - \mathbf{y}) = \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{D})$  with  $\mathbf{D} \rightarrow \mathbf{0}$ !

$$p(\mathbf{y}) = \int d\mathbf{x} \mathcal{N}(\mathbf{y}; \mathbf{t} + \mathbf{T}\mathbf{x}, \mathbf{D}) \mathcal{N}(\mathbf{x}; \mathbb{E}[\mathbf{x}], \mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D} \rightarrow \mathbf{0}$$

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$$p(\mathbf{y}) = \int d\mathbf{x} \mathcal{N}(\mathbf{y}; \mathbf{t} + \mathbf{T}\mathbf{x}, \mathbf{D}) \mathcal{N}(\mathbf{x}; \mathbb{E}[\mathbf{x}], \mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D} \rightarrow \mathbf{O}$$

$$= \mathcal{N}(\mathbf{y}; \mathbf{t} + \mathbf{T}\mathbb{E}[\mathbf{x}], \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top + \mathbf{D}) \quad \text{for } \mathbf{D} \rightarrow \mathbf{O}; \quad \text{product formula!}$$

**Also true if  $\dim(\mathbf{x}) \neq \dim(\mathbf{y})$ !**



# A popular model for object evolutions

## *Piecewise Constant White Acceleration Model*

Consider state vectors:  $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$  (position, velocity)

For known  $\mathbf{x}_{k-1}$  and without external influences we have with  $\Delta T_k = t_k - t_{k-1}$ :

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{I} & \Delta T_k \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1} \mathbf{x}_{k-1}, \quad \text{see blackboard!}$$

Assume during the interval  $\Delta T_k$  a constant acceleration  $\mathbf{a}_k$  causing the state evolution:

$$\begin{pmatrix} \frac{1}{2} \Delta T_k^2 \mathbf{I} \\ \Delta T_k \mathbf{I} \end{pmatrix} \mathbf{a}_k =: \mathbf{G}_k \mathbf{a}_k, \quad \text{linear transform!}$$

Let  $\mathbf{a}_k$  be a Gaussian RV with pdf:  $p(\mathbf{a}_k) = \mathcal{N}(\mathbf{a}_k; \mathbf{o}, \Sigma_k^2 \mathbf{I})$ , we therefore have:

$$p(\mathbf{G}_k \mathbf{a}_k) = \mathcal{N}(\mathbf{G}_k \mathbf{a}_k; \mathbf{o}, \Sigma_k^2 \mathbf{G}_k \mathbf{G}_k^\top).$$

Therefore:  $p(\mathbf{x}_k | \mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_k; \mathbf{F}_{k|k-1} \mathbf{x}_{k-1}, \mathbf{D}_{k|k-1})$  with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4} \Delta T_k^4 \mathbf{I} & \frac{1}{2} \Delta T_k^3 \mathbf{I} \\ \frac{1}{2} \Delta T_k^3 \mathbf{I} & \Delta T_k^2 \mathbf{I} \end{pmatrix}$$

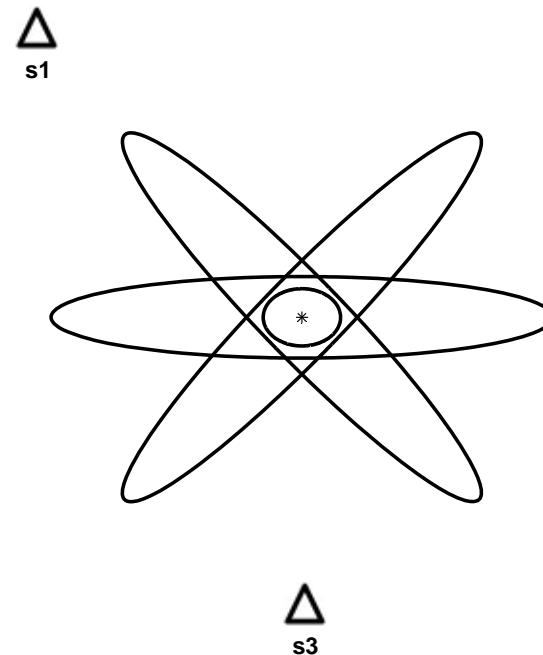
# Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by  $N$  sensors, we naïvely expect an improvement in accuracy  $\propto 1/\sqrt{N}$ .

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*a closer look:* The error of each measurement  $z_i$  is described by a related measurement error *covariance matrix*  $\mathbf{R}_i$  ('error ellipsoids'). In 2 dimensions:



$\mathbf{R}_i$  can strongly depend on the underlying sensor-to-target geometry!

# More Realistic: Range, Azimuth Measurements

- measurements in polar coordinates:

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \text{ measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, r, \varphi \text{ independent}$$

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Likelihood function in polar coordinates:

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- **What is the likelihood function in Cartesian coordinates?**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos \varphi_k \\ \sin \varphi_k \end{pmatrix}$$

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- **in Cartesian coord.: expand around**  $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$ :

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos \varphi_k \\ \sin \varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T} (\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

**constant and linear term of a Taylor series only, blackboard!**



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$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos \varphi_{k|k-1} & -r_{k|k-1} \sin \varphi_{k|k-1} \\ \sin \varphi_{k|k-1} & r_{k|k-1} \cos \varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

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- **in Cartesian coord.: expand around  $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$ :**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos \varphi_k \\ \sin \varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T} (\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos \varphi_{k|k-1} & -r_{k|k-1} \sin \varphi_{k|k-1} \\ \sin \varphi_{k|k-1} & r_{k|k-1} \cos \varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **affine transform of GAUSSIAN random variables:**

$$\mathcal{N}(\mathbf{z}; \mathbf{x}, \mathbf{R}) \xrightarrow{\mathbf{z}' = \mathbf{t} + \mathbf{T}\mathbf{z}} \mathcal{N}(\mathbf{z}'; \mathbf{t} + \mathbf{T}\mathbf{x}, \mathbf{T}\mathbf{R}\mathbf{T}^\top)$$

# More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \text{ measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around  $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$ :**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos \varphi_k \\ \sin \varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T} (\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos \varphi_{k|k-1} & -r_{k|k-1} \sin \varphi_{k|k-1} \\ \sin \varphi_{k|k-1} & r_{k|k-1} \cos \varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}}_{\text{rotation } D_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } S_r}$$

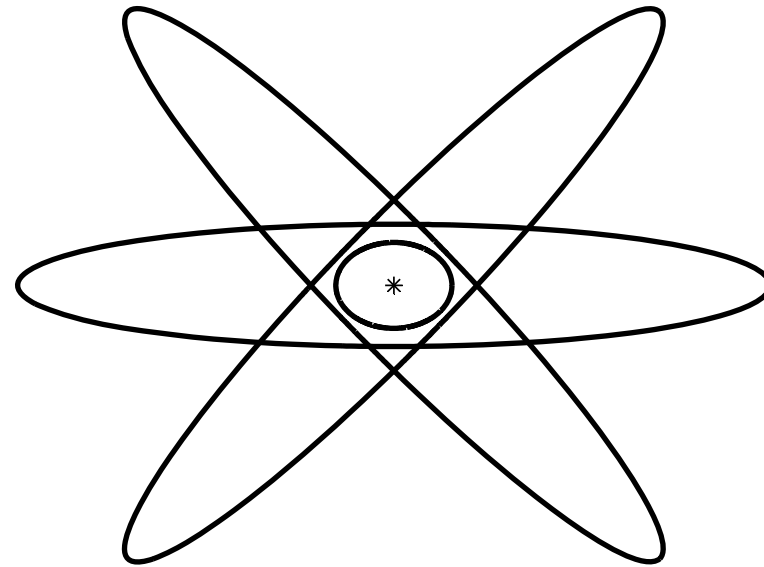
- **Cartesian error covariance (time dependent):**

$$\mathbf{T} \mathbf{R} \mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r \mathbf{R} \mathbf{S}_r \mathbf{D}_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r \sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top$$

- **sensor fusion: sensor-to-target-geometry enters into  $\mathbf{T} \mathbf{R} \mathbf{T}^\top$**

△  
s1

△  
s2



△  
s3

**sensor fusion: sensor-to-target-geometry enters into  $\mathbf{TRT}^T$**

# $S_k$ Sensors Producing Target Measurement at the Same Time

One possibility:

$$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \dots, \mathbf{R}_k^{S_k}]$$

# $S_k$ Sensors Producing Target Measurement at the Same Time

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$$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \dots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that  $\mathbf{H}_k^i = \mathbf{H}_k, i = 1, \dots, S_k$ :

$$\begin{aligned} p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) &= p(\mathbf{z}_k^1 | \mathbf{x}_k) p(\mathbf{z}_k^2 | \mathbf{x}_k) && \text{independent sensors} \\ &= \mathcal{N}(\mathbf{z}_k^1; \mathbf{H}_k \mathbf{x}_k, \mathbf{R}_k^1) \mathcal{N}(\mathbf{z}_k^2; \mathbf{H}_k \mathbf{x}_k, \mathbf{R}_k^2) \end{aligned}$$

# $S_k$ Sensors Producing Target Measurement at the Same Time

One possibility: 
$$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \dots, \mathbf{R}_k^{S_k}]$$

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# A Useful Product Formula for GAUSSIANS

$$\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{x}, \mathbf{R}) \mathcal{N}(\mathbf{x}; \mathbf{y}, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z}; \mathbf{H}\mathbf{y}, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x}; \mathbf{y} + \mathbf{W}\boldsymbol{\nu}, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x}; \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}), \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$



# $S_k$ Sensors Producing Target Measurement at the Same Time

One possibility: 
$$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \dots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that  $\mathbf{H}_k^i = \mathbf{H}_k, i = 1, \dots, S_k$ :

$$\begin{aligned} p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) &= p(\mathbf{z}_k^1 | \mathbf{x}_k) p(\mathbf{z}_k^2 | \mathbf{x}_k) && \text{independent sensors} \\ &= \mathcal{N}(\mathbf{z}_k^1; \mathbf{H}_k \mathbf{x}_k, \mathbf{R}_k^1) \mathcal{N}(\mathbf{z}_k^2; \mathbf{H}_k \mathbf{x}_k, \mathbf{R}_k^2) \\ &= \mathcal{N}(\mathbf{H}_k \mathbf{x}_k; \mathbf{z}_k^1, \mathbf{R}_k^1) \mathcal{N}(\mathbf{z}_k^2; \mathbf{H}_k \mathbf{x}_k, \mathbf{R}_k^2) \\ &\propto \mathcal{N}(\mathbf{H}_k \mathbf{x}_k; \underbrace{\mathbf{R}_k (\mathbf{R}_k^1)^{-1} \mathbf{z}_k^1 + \mathbf{R}_k^2)^{-1} \mathbf{z}_k^2}_{=\mathbf{z}_k}, \underbrace{(\mathbf{R}_k^1)^{-1} + \mathbf{R}_k^2)^{-1}}_{=\mathbf{R}_k}) \\ &= \mathcal{N}(\mathbf{z}_k; \mathbf{H}_k \mathbf{x}_k, \mathbf{R}_k) \end{aligned}$$

## Exercise 3.3

Generalize to the case  $S_k > 2$  (induction argument)!

**One possible fusion strategy:** Create a single *effective measurement* by preprocessing of the individual measurements!

$$\mathbf{z}_k = \mathbf{R}_k \sum_{s=1}^{S_k} (\mathbf{R}_k^s)^{-1} \mathbf{z}_k^s \quad \text{weighted arithmetic mean of measurements}$$

$$\mathbf{R}_k = \left( \sum_{s=1}^{S_k} (\mathbf{R}_k^s)^{-1} \right)^{-1} \quad \text{harmonic mean of measurement covariances}$$

**A typical structure for fusion equations!**