Summary: BAYESian (Multi-) Sensor Tracking

- **Basis:** In the course of time one or several sensors produce measurements of targets of interest. Each target is characterized by its current state vector, being expected to change with time.

- **Objective:** Learn as much as possible about the individual target states at each time by analyzing the 'time series' which is constituted by the sensor data.

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- **Problem:** imperfect sensor information: inaccurate, incomplete, and eventually ambiguous. Moreover, the targets' temporal evolution is usually not well-known.
- **Approach:** Interpret measurements and target vectors as random variables (RVs). Describe by probability density functions (pdf) what is known about them.
- **Solution:** Derive iteration formulae for calculating the pdfs! Develop a mechanism for initiation! By doing so, exploit all background information available! Derive state estimates from the pdfs along with appropriate quality measures!

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Elements for multisensor situation pictures: tracks of temporally evolving objects

Which object properties are of interest? $\rightarrow$ *state* X_k *at time* t_k

- road-moving vehicle: odometer count x_k $\qquad X_k = (x_k, \dot{x}_k)$
- position, speed, acceleration: $\qquad X_k = (\mathbf{r}_k, \dot{\mathbf{r}}_k, \ddot{\mathbf{r}}_k)$
- joint state of several objects: $\qquad X_k = (\mathbf{x}_k^1, \mathbf{x}_k^2, \ldots)$
- attributes, e.g. radar cross section $x_k \in \mathbb{R}^+$: $X_k = (\mathbf{x}_k, x_k)$
- maneuvering phase, object class $i_k \in \mathbb{N}$: $\qquad X_k = (\mathbf{x}_k, i_k)$

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Elements for multisensor situation pictures: tracks of temporally evolving objects

Which object properties are of interest? $\rightarrow$ *state* X_k *at time* t_k

How to lean states X_k? $\rightarrow$ from *sensor data* $Z^k = \{Z_k, Z^{k-1}\}$, *context*

How to imprecise information? $\rightarrow$ *e.g. by conditional pdfs* $p(X_k|Z^k)$

What means “learning” in this context? $\rightarrow$ *iterative calculation of* $p(X_k|Z^k)$

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The general tracking equations

Prediction

$$p(X_k|Z^{k-1}) = \int dX_{k-1}\, \underbrace{p(X_k|X_{k-1})}_{\text{evolution}}\, \underbrace{p(X_{k-1}|Z^{k-1})}_{\text{filtering } t_{k-1}}$$

filtering

$$p(X_k|Z^k) = \frac{p(Z_k|X_k)\, p(X_k|Z^{k-1})}{\int dX_k\, \underbrace{p(Z_k|X_k)}_{\text{sensor model}}\, \underbrace{p(X_k|Z^{k-1})}_{\text{prediction}}}$$

retrodiction

$$p(X_l|Z^k) = \int dX_{l+1}\, \frac{\overbrace{p(X_{l+1}|X_l)}^{\text{evolution}}\, \overbrace{p(X_l|Z^l)}^{\text{filtering } t_l}}{\underbrace{p(X_{l+1}|Z^l)}_{\text{prediction } t_{l+1}}}\, \underbrace{p(X_{l+1}|Z^k)}_{\text{retrodiction } t_{l+1}}$$

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Elements for situation pictures: tracks of temporally evolving objects

Which object properties are of interest? $\rightarrow$ *state* X_k *at time* t_k

How to lean states X_k? $\rightarrow$ from *sensor data* $Z^k = \{Z_k, Z^{k-1}\}$, *context*

How to imprecise information? $\rightarrow$ *e.g. by conditional pdfs* $p(X_k|Z^k)$

What means "learning" in this context? $\rightarrow$ *iterative calculation of* $p(X_k|Z^k)$

What is needed for this? $\rightarrow$ *evolution / sensor models* $p(X_k|X_{k-1})$, $p(Z_k|X_k)$

How to initiate / terminate tracking processes? $\rightarrow$ *sequential decisions*

Why is 'target tracking' a key function?

infer secondary quantities from incomplete measurement data.

Eliminate fluctuating false return background (clutter).

Create a time basis for classification from attribute data.

...

'Shape' of objects / object groups relevant in many applications.

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Track-based inference of object properties

- **Velocity history:** vehicle, helicopter, plane
- **Acceleration history:** threat: no under-wing weapons
- **Rare events:** truck by night on dirt road near a border
- **Object interrelations:** resulting from formation, convoy
- **Object sources / sinks:** classification by origin / designation
- **Classification:** road-moving vehicle, ‘on-road’ $\rightarrow$ ‘off-road’

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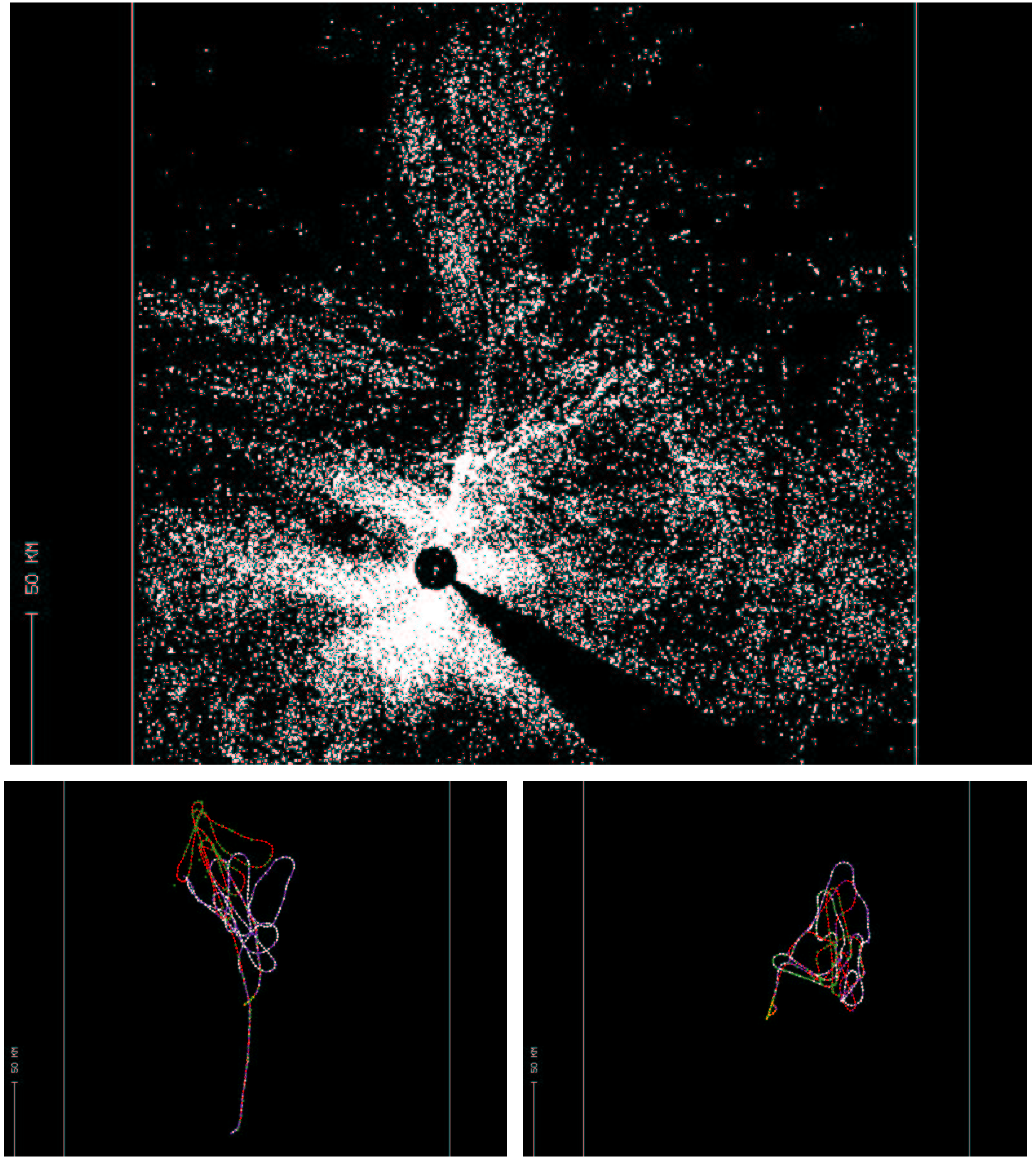
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How to deal with probability density functions?

- pdf $p(x)$: Extract *probability statements* about the RV x by integration!
- naïvely: *positive* and *normalized* functions ($p(x) \geq 0$, $\int dx\, p(x) = 1$)
- *conditional pdf* $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on y on RV x?
- *marginal density* $p(x) = \int dy\, p(x,y) = \int dy\, p(x|y)\, p(y)$: Enter y!
- Bayes: $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx\, p(y|x)p(x)}$: $p(x|y) \leftarrow p(y|x), p(x)$!

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Recapitulation: The Multivariate GAUSSian Pdf

- *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$
- *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

  $q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

- *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\, \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top \mathbf{P}^{-1}(x-\mathbf{x})}$$

- GAUSS*ian Mixtures:* $p(x) = \sum_i p_i\, \mathcal{N}(x;\, \mathbf{x}_i,\, \mathbf{P}_i)$ (weighted sums)

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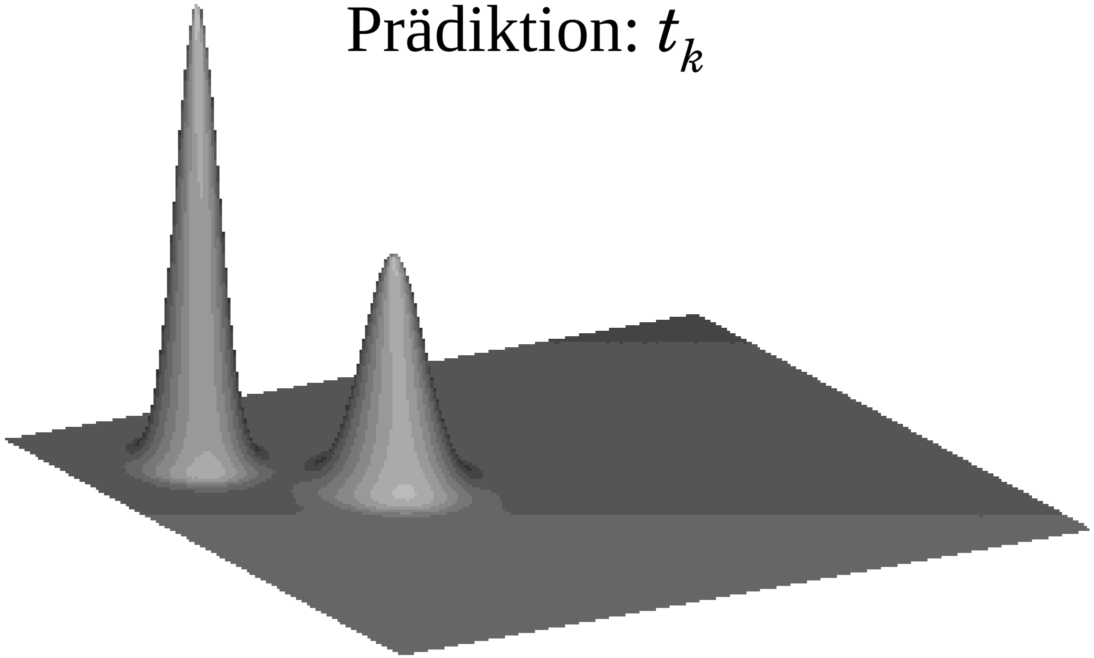

Exploit imprecise knowledge on the dynamical behavior of the object.

$$\underbrace{p(\mathbf{x}_k|\mathcal{Z}^{k-1})}_{\text{prediction}} = \int d\mathbf{x}_{k-1}\ \underbrace{\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}\mathbf{x}_{k-1},\ \mathbf{D}\big)}_{\text{dynamics}}\ \underbrace{\mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}_{k-1|k-1},\ \mathbf{P}_{k-1|k-1}\big)}_{\text{old knowledge}}.$$

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A Useful Product Formula for GAUSSians

$$\mathcal{N}(\mathbf{z};\, \mathbf{F}\mathbf{x},\, \mathbf{D})\; \mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{F}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}}\; \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top)$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{F}\mathbf{y}, \quad \mathbf{S} = \mathbf{F}\mathbf{P}\mathbf{F}^\top + \mathbf{D}, \quad \mathbf{W} = \mathbf{P}\mathbf{F}^\top\mathbf{S}^{-1}.$$

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

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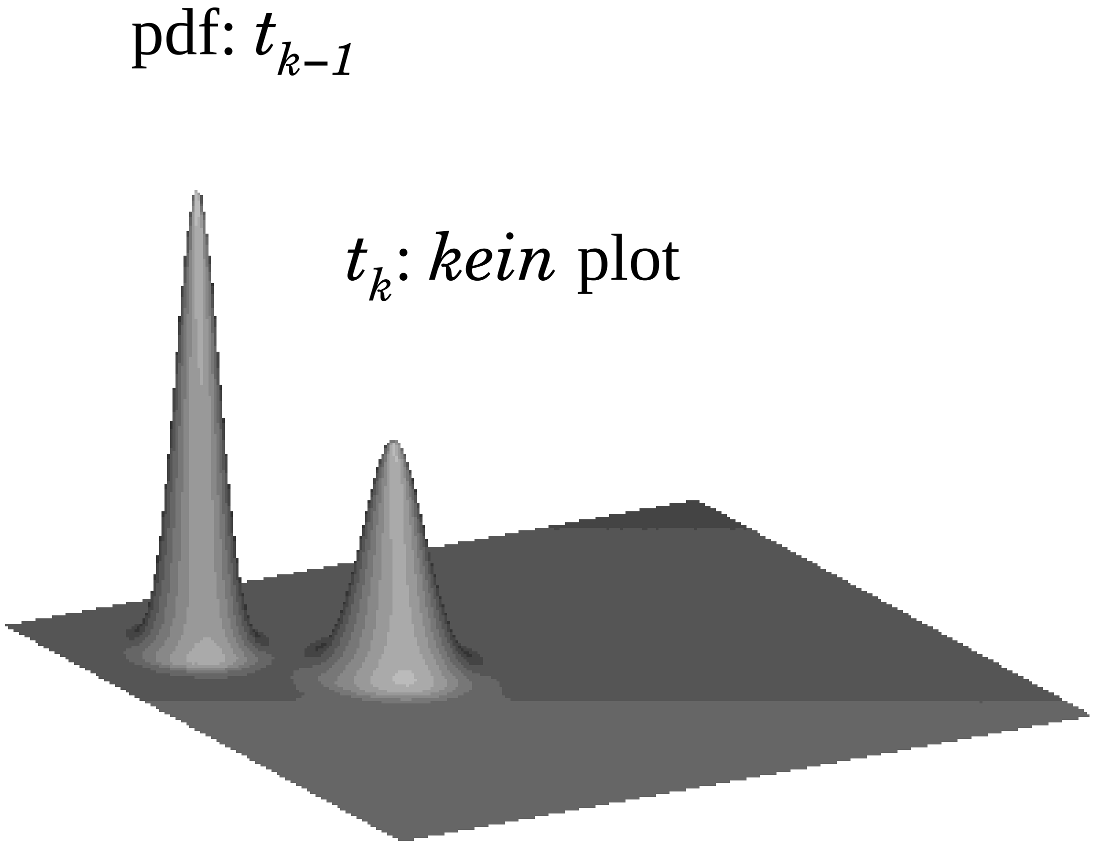

missing sensor detection: ‘data processing’ = prediction
(not always: exploitation of ‘negative’ sensor evidence)

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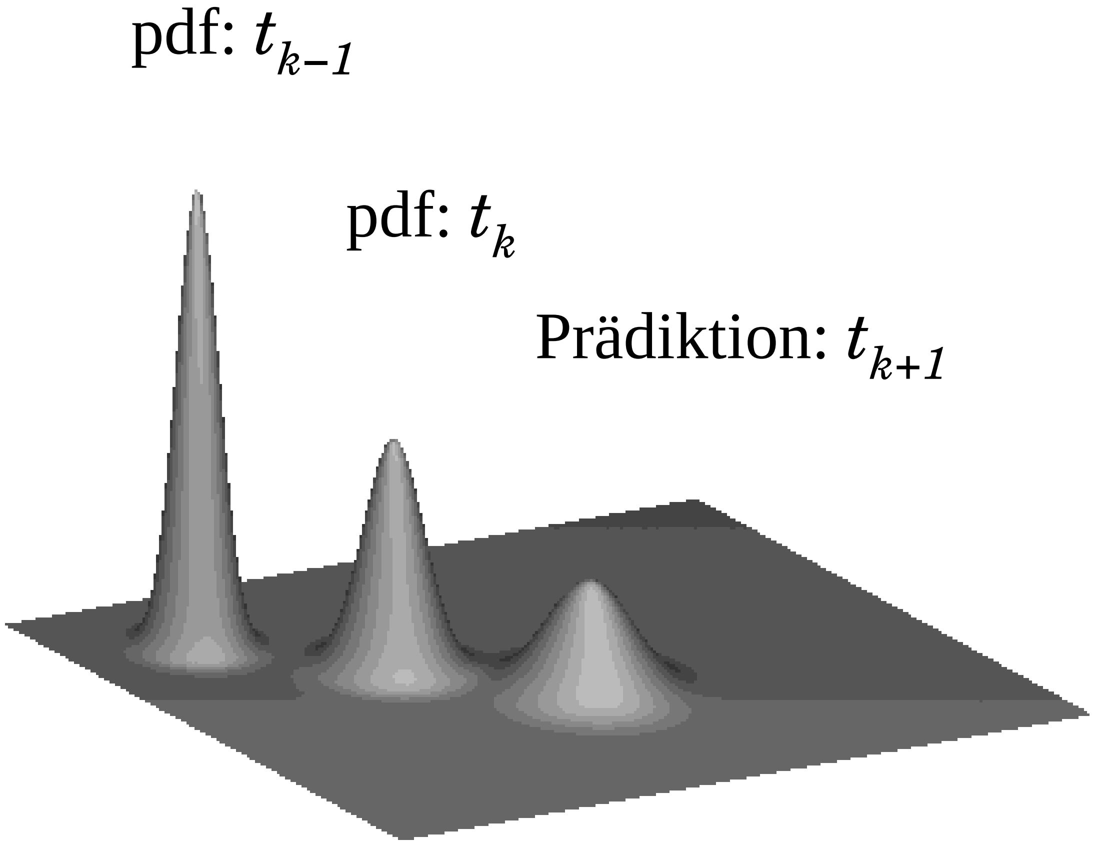
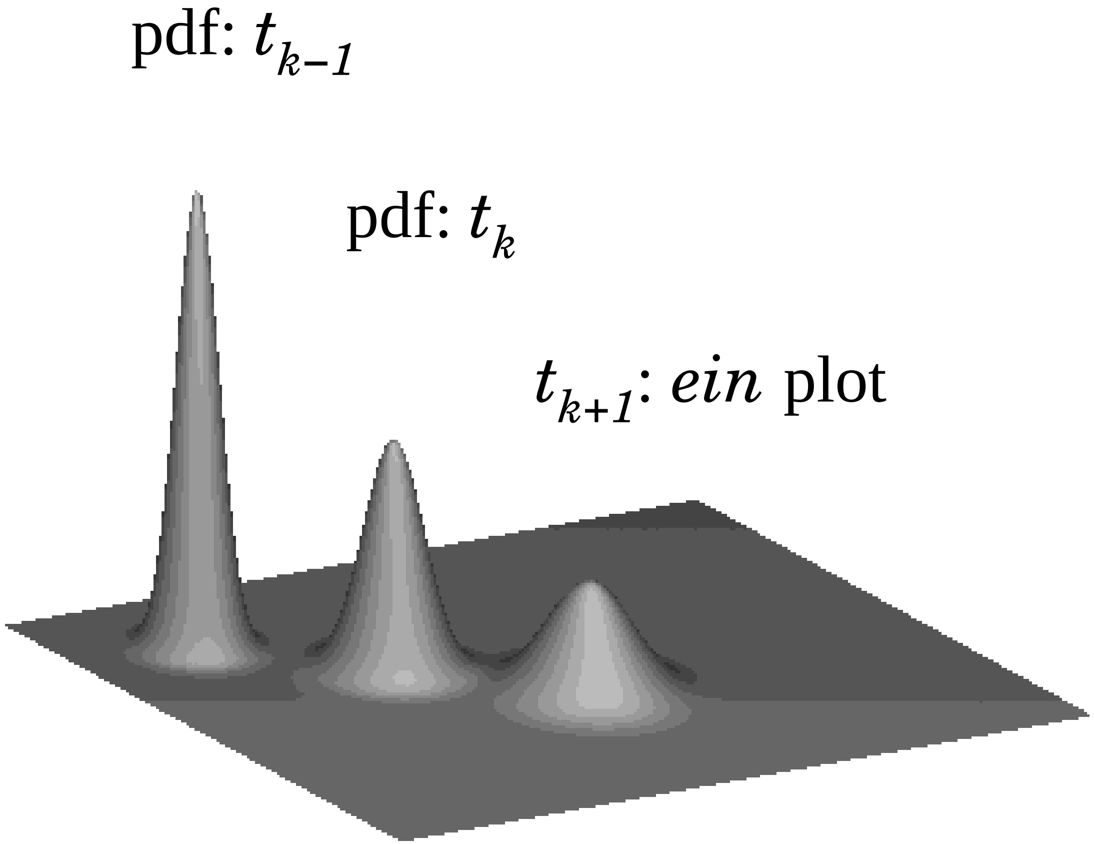
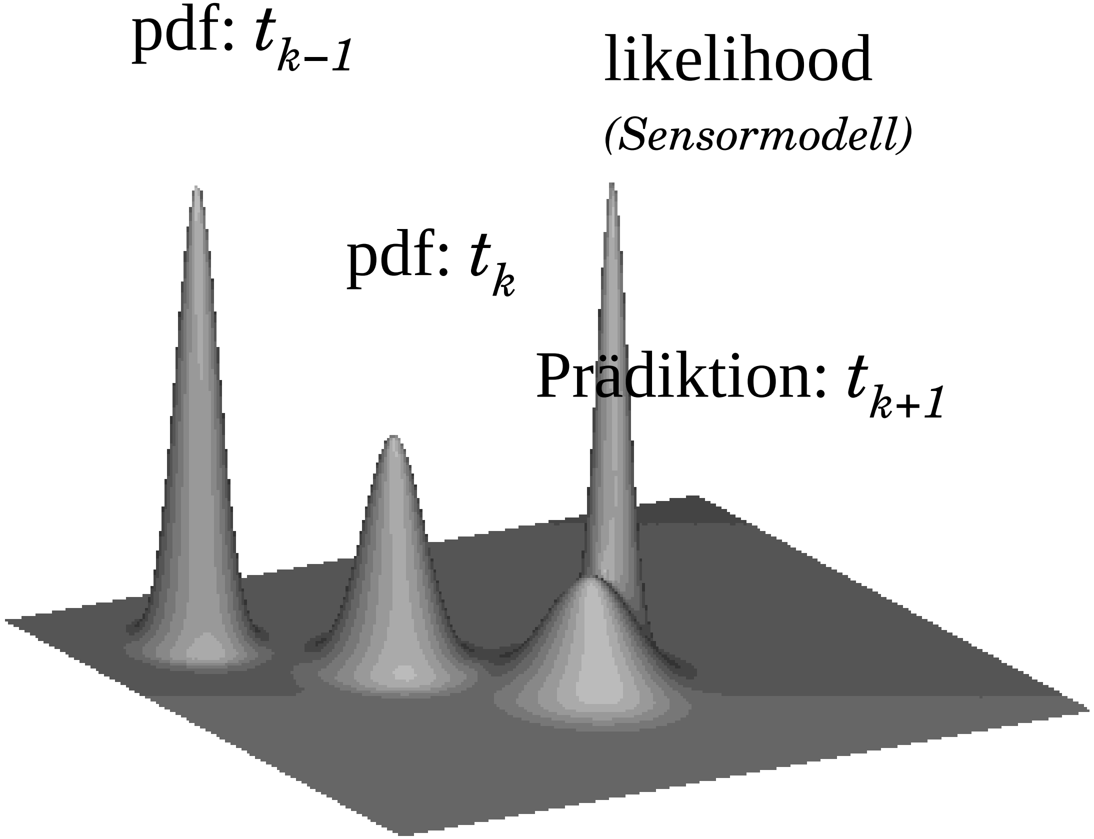


missing sensor information: increasing knowledge dissipation



sensor information on the kinematical object state



BAYES' formula:

$$\underbrace{p(\mathbf{x}_{k+1}|\mathcal{Z}^{k+1})}_{\text{new knowledge}} = \frac{p(\mathbf{z}_{k+1}|\mathbf{x}_{k+1})\, p(\mathbf{x}_{k+1}|\mathcal{Z}^k)}{\int d\mathbf{x}_{k+1}\, p(\underbrace{\mathbf{z}_{k+1}}_{\text{plot}}|\mathbf{x}_{k+1})\, \underbrace{p(\mathbf{x}_{k+1}|\mathcal{Z}^k)}_{\text{prediction}}}$$

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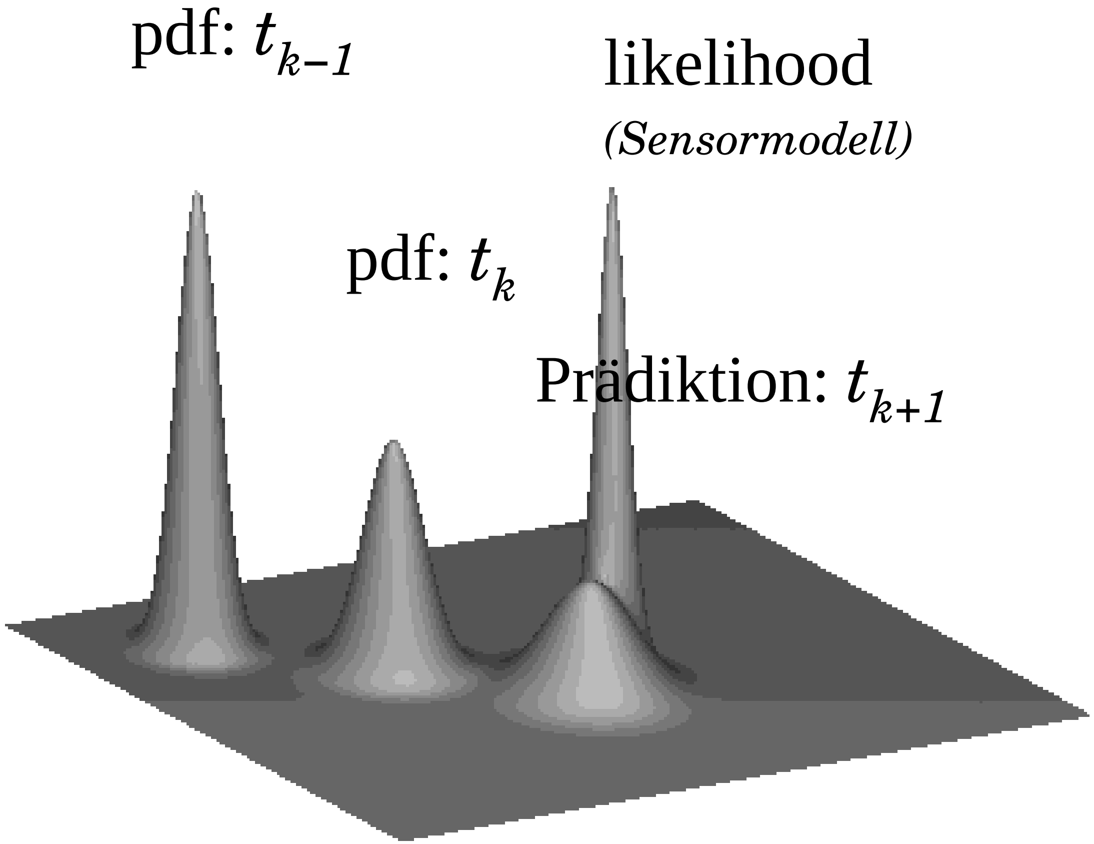

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$$\underbrace{p(\mathbf{x}_{k+1}|\mathcal{Z}^{k+1})}_{\text{new knowledge}} = \frac{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big)\ \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}{\int d\mathbf{x}_{k+1}\ \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big)\ \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}_{\text{prediction}}}$$

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A Useful Product Formula for Gaussians

$$\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R}\big)\; \mathcal{N}\big(\mathbf{x};\, \mathbf{y},\, \mathbf{P}\big) = \underbrace{\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S}\big)}_{\text{independent of } \mathbf{x}}\; \mathcal{N}\big(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top\big)$$

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$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering:

$$\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$

$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$ 'KALMAN gain matrix'

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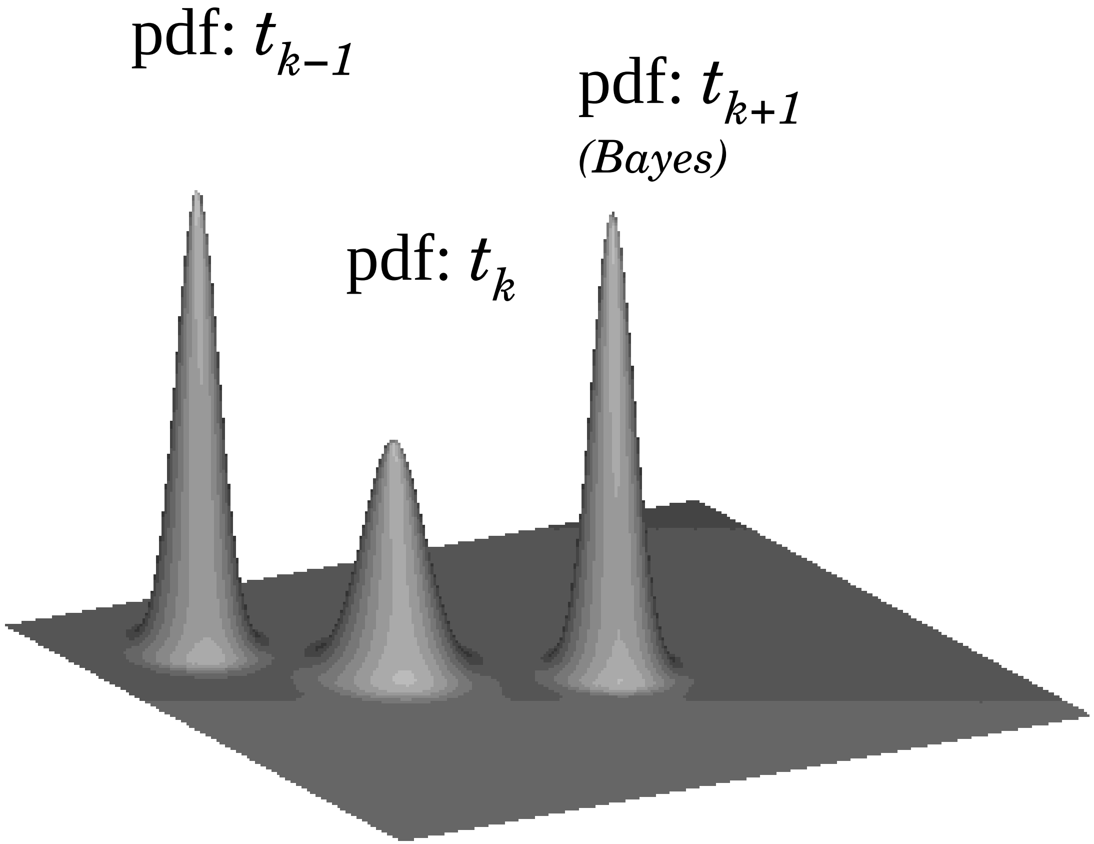

filtering = sensor data processing

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- *certain knowledge* on x: $p(x) = \delta(x - y)$ '=' $\lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-y)^2}{\sigma^2}}$

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Why is ‘tracking’ a key function for ‘understanding’ a situation?

infer secondary quantities from incomplete measurement data.

Eliminate fluctuating false return background (clutter).

Create a time basis for classification from attribute data.

. . .

‘Shape’ of objects / object groups relevant in many applications.

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modeling: sensor data produced by extended objects

- **actual measurement errors of individual scattering centers unimportant**
- **the 'message' of individual plots is dominated by the object extension**
- **individual plots to be interpreted as measurements of the object center**
- **related 'measurement error' proportional to extension *to be estimated***

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object extension: ‘covariance-type’ matrices

- state augmentation by a random matrix
  - object extension exceeding resolution
  - closely spaced vehicle convoys
  - collectively moving object clouds

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Elements for Situation Pictures: Tracks of Time-varying Objects

Which object properties are of interest? $\rightarrow$ ***state*** X_k ***at time*** t_k

- road-moving vehicle: odometer count x_k $\quad X_k = (x_k, \dot{x}_k)$
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- maneuvering phase, object class $i_k \in \mathbb{N}$: $\quad X_k = (\mathbf{x}_k, i_k)$
- object shape: SPD random matrices $\mathbf{X}_k$ $\quad X_k = (\mathbf{x}_k, \mathbf{X}_k)$

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extended objects: simplified description

- **kinematical state at time t_k: x_k = (position, velocity, . . .)**
- **object extension at time t_k: approximately by an ellipse**
- **size: volume, shape: ratio of semi-axes, spatial orientation**
- **extension: SPD matrix X_k (Symmetric, Positively Definite)**

augmented state: kinematical state *vector* x_k, extension *matrix* X_k

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Generalize BAYESian tracking to extended objects.

n_k plots $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$ at time t_k, accumulated data $\mathcal{Z}^k = \{Z_k, n_k, \mathcal{Z}^{k-1}\}$

The conditional pdf $p(\mathrm{x}_k, \mathrm{X}_k|\mathcal{Z}^k)$ describes what is known about the extended object state $\mathrm{x}_k, \mathrm{X}_k$ based on all sensor data up to time t_k.

'extended object tracking': *iterative* calculation of $p(\mathrm{x}_k, \mathrm{X}_k|\mathcal{Z}^k)$.

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Generalize BAYESian tracking to extended objects.

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- $p(\mathrm{x}_k, \mathrm{X}_k|\mathcal{Z}^k) = p(\mathrm{x}_k|\mathrm{X}_k, \mathcal{Z}^k)\, p(\mathrm{X}_k|\mathcal{Z}^k)$ **extended object 'track'**
- $p(Z_k, n_k|\mathrm{x}_k, \mathrm{X}_k)$ **sensor output to be processed, i.e. the likelihood**
- $p(\mathrm{x}_k|\mathcal{Z}^k) = \int d\mathrm{X}_k\, p(\mathrm{x}_k, \mathrm{X}_k|\mathcal{Z}^k)$ **kinematics of the extended object**

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Generalize BAYESian tracking to extended objects.

n_k plots $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$ at time t_k, accumulated data $\mathcal{Z}^k = \{Z_k, n_k, \mathcal{Z}^{k-1}\}$

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$$p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^k) = \frac{p(Z_k, n_k|\mathbf{x}_k, \mathbf{X}_k)\, p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k d\mathbf{X}_k\, p(Z_k, n_k|\mathbf{x}_k, \mathbf{X}_k)\, p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^{k-1})}$$

- $p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^k) = p(\mathbf{x}_k|\mathbf{X}_k, \mathcal{Z}^k)\, p(\mathbf{X}_k|\mathcal{Z}^k)$ **extended object 'track'**

  $p(\mathbf{x}_k|\mathbf{X}_k, \mathcal{Z}^k)$: **assume Gaussian density**

  $p(\mathbf{X}_k|\mathcal{Z}^k)$: **assume inverse Wishart density, details later!**

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Characterize an object by *quantitatively describable* properties: object state

Examples:

- object position x on a strait line: $x \in \mathbb{R}$
- kinematic state $\mathbf{x} = (\mathbf{r}^\top, \dot{\mathbf{r}}^\top, \ddot{\mathbf{r}}^\top)^\top$, $\mathbf{x} \in \mathbb{R}^9$

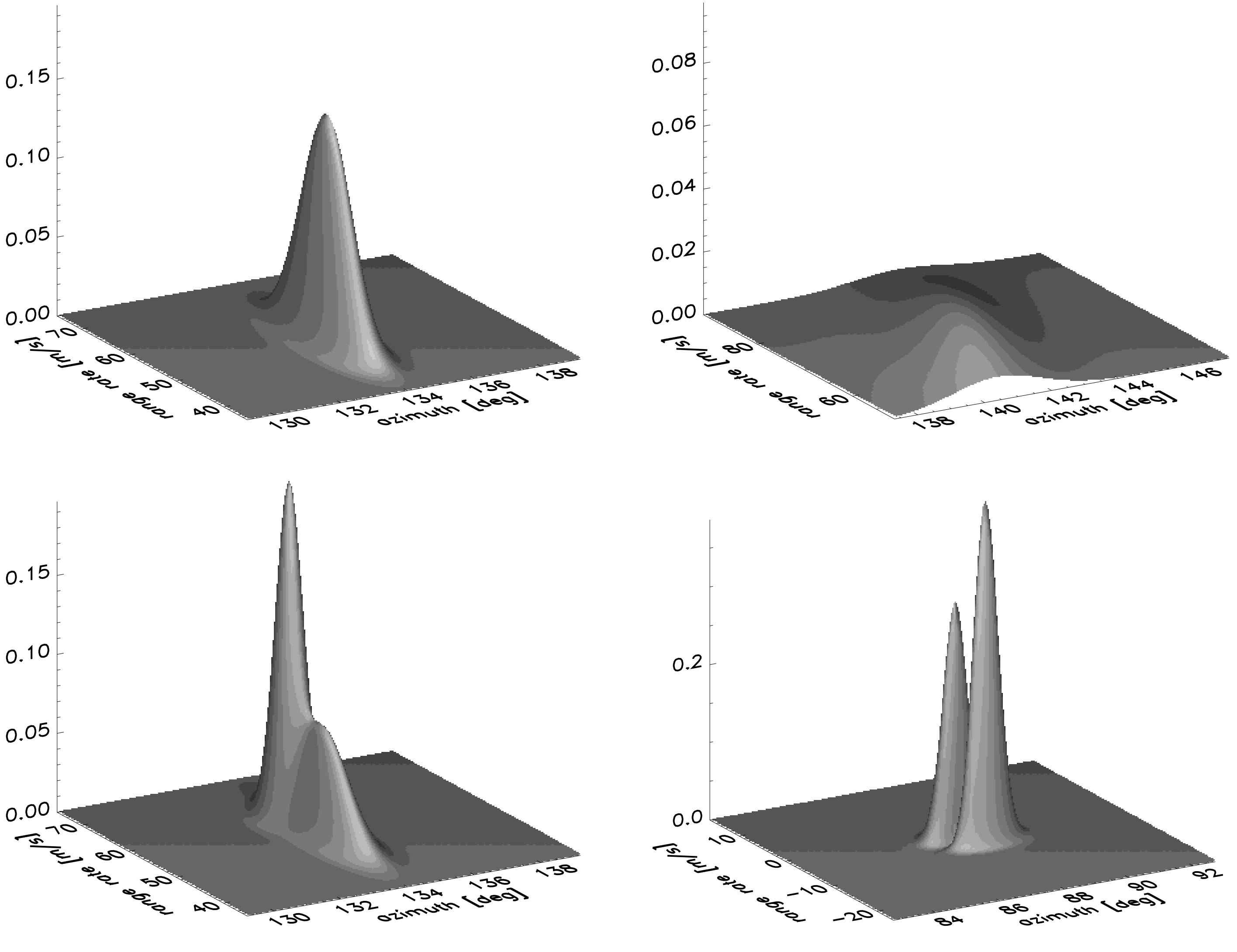
  position $\mathbf{r} = (x, y, z)^\top$, velocity $\dot{\mathbf{r}}$, acceleration $\ddot{\mathbf{r}}$
- joint state of two objects: $\mathbf{x} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top)^\top$
- kinematic state $\mathbf{x}$, object extension $\mathbf{X}$

  z.B. ellipsoid: symmetric, positively definite matrix
- kinematic state $\mathbf{x}$, object class class

  z.B. bird, sailing plane, helicopter, passenger jet, ...

Learn unknown object states from imperfect measurements and
describe by functions $p(\mathbf{x})$ imprecise knowledge mathematically precisely!

Interpret unknown object states as *random variables*, x [1D] or $\mathbf{x}$, $\mathbf{X}$ [vector / matrix variate]), characterized by corresponding *probability density functions* (pdf).



The concrete shape of the pdf $p(\mathbf{x})$ contains the full knowledge on $\mathbf{x}$!

Information on a random variable (RV) can be extracted by integration from the corresponding pdf. !

at present: one dimensional case:

How probable is it that $x \in (a, b) \subseteq \mathbb{R}$ ***holds?***

Answer: $P\{x \in (a, b)\} = \int_a^b dx\, p(x) \quad \Rightarrow \quad p(x) \geq 0$

in particular: $P\{x \in \mathbb{R}\} = \int_{-\infty}^{\infty} dx\, p(x) = 1$ (normalzation)

intuitive interpretation: "*the object is somewhere in* $\mathbb{R}$"

loosely: $p(x)\, dx$ **is probabity for** x **having a value between** x **and** $x + dx$

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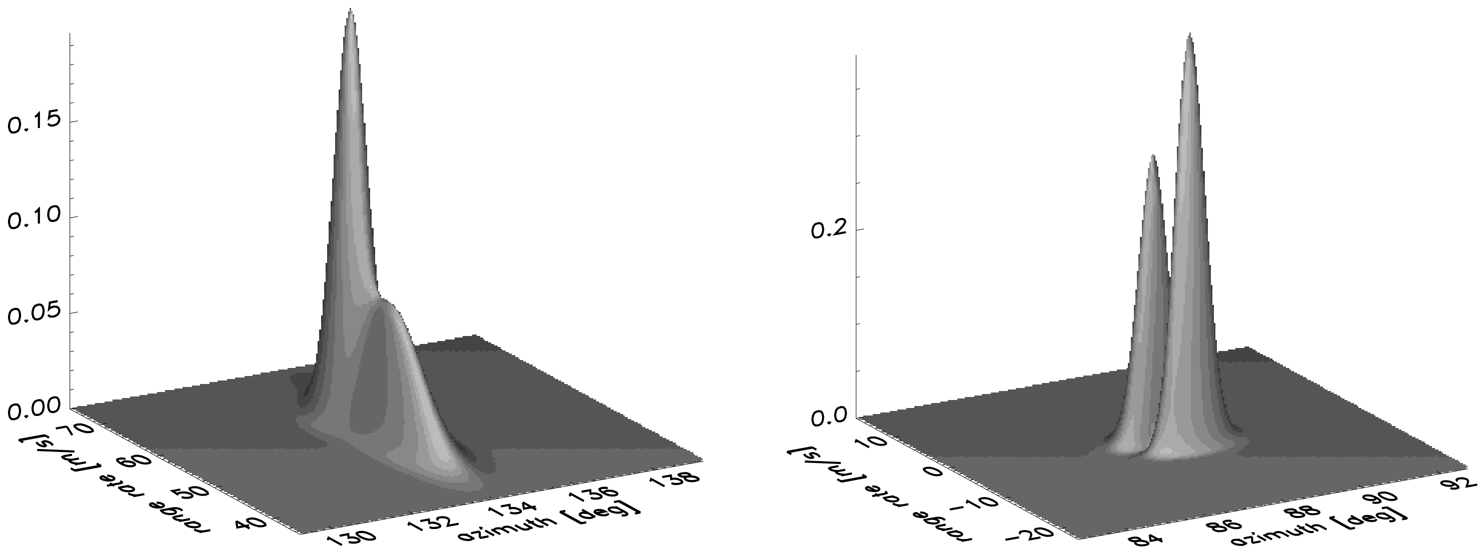
How to characterize the properties of a pdf?

specifically: How to associate a single “expected” value to a RV?

The maximum of the pdf is sometimes but not always useful!



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How to characterize the properties of a pdf?

specifically: How to associate a single "expected" value to a RV?

The maximum of the pdf is sometimes but not always useful! ($\rightarrow$ examples)

instead: Calculate the centroid of the pdf!

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, p(x) = \bar{x}$$ "expectation value"

more generally: Consider functions $g : x \mapsto g(x)$ of the RV x!

$$\mathbb{E}[g(x)] = \int_{-\infty}^{\infty} dx\, g(x)\, p(x),$$ "expectation value of the *observable* g"

Example: Consider the observable $\frac{1}{2}mx^2$ (kinetic energy, x = speed)

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An important observable: the “error” of an estimate

- **Quality:** How useful is an expectation value $\bar{x} = \mathbb{E}[x]$?

Consider special obervables as distance measure:

$$g(x) = |x - \bar{x}| \quad \text{oder} \quad g(x) = (x - \bar{x})^2$$

quadratic measures: computationally more comfortable!

‘expected error’ of the expectation value $\bar{x}$:

$$\boxed{\mathbb{V}[x] = \mathbb{E}[(x - \bar{x})^2],\ \sigma_x = \sqrt{\mathbb{V}[x]}}$$

variance, standard deviation

Exercise 2.1

Show that $\boxed{\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2}$ holds.

Expectation value of the observable x^2 also called “2nd moment” of the pdf of x.

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Calculate expectation and variance of the uniform density of a RV $x \in \mathbb{R}$ in the intervall $[a, b]$.

Exercise 2.2

$$p(x) = \mathcal{U}(\underbrace{x}_{\text{ZV}};\ \underbrace{a, b}_{\text{Parameter}}) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

!

Pdf correctly normalized? $\int_{-\infty}^{\infty} dx\, \mathcal{U}(x; a, b) = \frac{1}{b-a} \int_a^b dx = 1$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{U}(x; a, b) = \frac{b+a}{2}$$

$$\mathbb{V}[x] = \frac{1}{b-a} \int_a^b dx\, x^2 - \mathbb{E}[x]^2 = \frac{1}{12}(b-a)^2$$

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Important example: x *normally distributed* over $\mathbb{R}$ (Gauss)

- *wanted:* probabilities concentrated around μ
- quadratic distance: $||x-\mu||^2 = \frac{1}{2}(x-\mu)^2/\sigma^2$ (mathematically convenient!)
- Parameter σ is a measure of the "width" of the pdf: $||\sigma||^2 = \frac{1}{2}$
- for 'large' distances, i.e. $||x-\mu||^2 \gg \frac{1}{2}$, the pdf shall decay quickly.
- simplest approach: $\tilde{p}(x) = \mathrm{e}^{-||x-\mu||^2}$ $\quad$ ($> 0\ \forall x \in \mathbb{R}$, normalization?)
- Normalized for $p(x) = \tilde{p}(x) / \int_{-\infty}^{\infty} dx\, \tilde{p}(x)$!

Formula collection delivers: $\int_{-\infty}^{\infty} dx\, \tilde{p}(x) = \sqrt{2\pi}\sigma$

An admissible pdf with the required properties is obviously given by:

$$\mathcal{N}(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

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Exercise 2.3

Show for the Gaussian density $p(x) = \mathcal{N}(x; \mu, \sigma)$:

$$\boxed{\mathbb{E}[x] = \mu, \;\; \mathbb{V}[x] = \sigma^2}$$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{N}(x; \mu, \sigma) = \mu$$

$$\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2$$

Use substitution and partial integration!

Use $\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\frac{1}{2}x^2} = \sqrt{2\pi}$!

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