Refined Sensor Modeling: Resolution Phenomena

- **band/beam width, coherence:** e.g. for RADAR: α_r = 200 m, α_φ = 2°, $\alpha_{\dot{r}}$ = 2 m/s
- **irresolved measurement:** $\mathbf{H}_g\,\mathbf{x}_k = \frac{1}{2}\,\mathbf{H}(\mathbf{x}_k^1 + \mathbf{x}_k^2)$ "center of gravity"
- **resolution (qualitatively):**
  - · depending on target-sensor geometry, rel. orientation
  - · resolution capability in r, φ, $\dot{r}$ mutually independent
  - · very low resolution for: $\Delta r < \alpha_r$, $\Delta\varphi < \alpha_\varphi$, $\Delta\dot{r} < \alpha_{\dot{r}}$
  - · no resolution phenomena: $\Delta r \gg \alpha_r$, $\Delta\varphi \gg \alpha_\varphi$, $\Delta\dot{r} \gg \alpha_{\dot{r}}$
  - · small transient region between these domains

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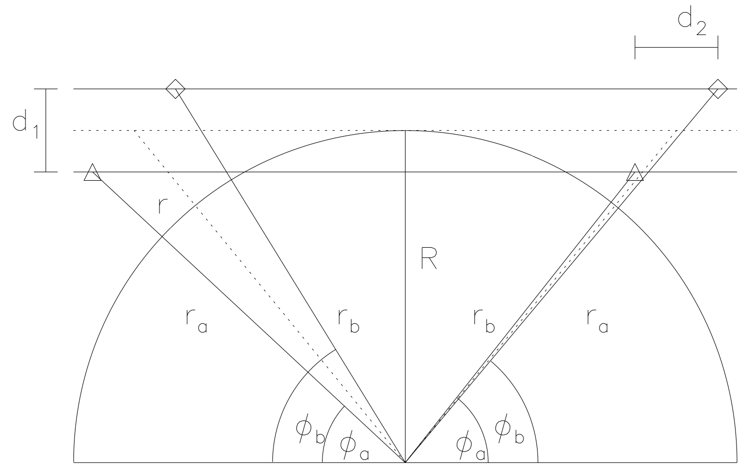
- **Echelon formation:**



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- **a simple resolution model:**

$$P_r(\Delta r, \Delta\varphi, \Delta\dot r) = 1 - P_u(\Delta r, \Delta\varphi, \Delta\dot r)$$

$$P_u = \mathrm{e}^{-\log 2(\frac{\Delta r}{\alpha_r})^2}\,\mathrm{e}^{-\log 2(\frac{\Delta\varphi}{\alpha_\varphi})^2}\,\mathrm{e}^{-\log 2(\frac{\Delta\dot r}{\alpha_{\dot r}})^2}$$

$$= |2\pi\mathbf{R}_u|^{\frac{1}{2}}\,\mathcal{N}\big(\mathrm{O};\,\mathbf{H}(\mathbf{x}_k^1 - \mathbf{x}_k^2),\,\mathbf{R}_u\big)$$

Interpretation hypotheses: $\ell(Z_k, n_k|\mathbf{x}_k) = \sum_{E_k} \ell(Z_k, n_k, E_k|\mathbf{x}_k)$

- E_k^{ii}: Objects irresolved, detected as a group, $\mathbf{z}_k^i \in Z_k$ being the plot:

$$\ell(Z_k, n_k, E_k^{ii}|\mathbf{x}_k) = \text{const.} \times P_u(\mathbf{x}_k)\,\mathcal{N}(\mathbf{z}_k^i; \mathbf{H}_k^g \mathbf{x}_k, \mathbf{R}_k^g)$$

$$= \text{const.}' \times \mathcal{N}\left(\begin{pmatrix} \mathbf{z}_k^i \\ \mathbf{o} \end{pmatrix}; \begin{pmatrix} \mathbf{H}_k^g \\ \mathbf{H}_u \end{pmatrix} \mathbf{x}_k, \begin{pmatrix} \mathbf{R}_k^g & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_u \end{pmatrix}\right)$$

Assuming E_k^{ii}, we process a (real) measurement $\mathbf{z}_k^i$ of the center $\frac{1}{2}\mathbf{H}(\mathbf{x}_k^1 + \mathbf{x}_k^2)$ and a (fictitious) measurement "zero" of the distance $\mathbf{H}(\mathbf{x}_k^1 - \mathbf{x}_k^2)$ between the targets. $\mathbf{R}_u$ defines the resolution capability.

Interpretation hypotheses: $\ell(Z_k, n_k|\mathbf{x}_k) = \sum_{E_k} \ell(Z_k, n_k, E_k|\mathbf{x}_k)$

- E_k^{ii}: Objects irresolved, detected as a group; $\mathbf{z}_k^i \in Z_k$ being the plot

$$\ell(Z_k, n_k, E_k^{ii}|\mathbf{x}_k) = \text{const.} \times \mathcal{N}\left(\begin{pmatrix} \mathbf{z}_k^i \\ \mathbf{o} \end{pmatrix}; \begin{pmatrix} \mathbf{H}_k^g \\ \mathbf{H}_u \end{pmatrix} \mathbf{x}_k, \begin{pmatrix} \mathbf{R}_k^g & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_u \end{pmatrix}\right)$$

- E_k^{00}: Objects neither resolved, nor detected; all plots are false

$$\ell(Z_k, n_k, E_k^{00}|\mathbf{x}_k) = P_u(\mathbf{x}_k)\,(1-P_D^u)\,\frac{p_F(n_k)}{|\text{FoV}|^{n_k}}$$

$$= \text{const.} \times \mathcal{N}\big(\mathbf{O};\, \mathbf{H}(\mathbf{x}_k^1 - \mathbf{x}_k^2),\, \mathbf{R}_u\big)$$

Assuming E_k^{00}, a fictitious zero-distance measurement is processed.

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$$\ell(Z_k, n_k, E_k^{ii}|\mathbf{x}_k) = \text{const.} \times \mathcal{N}\left(\begin{pmatrix}\mathbf{z}_k^i\\ \mathbf{o}\end{pmatrix}; \begin{pmatrix}\mathbf{H}_k^g\\ \mathbf{H}_u\end{pmatrix}\mathbf{x}_k, \begin{pmatrix}\mathbf{R}_k^g & \mathbf{O}\\ \mathbf{O} & \mathbf{R}_u\end{pmatrix}\right)$$

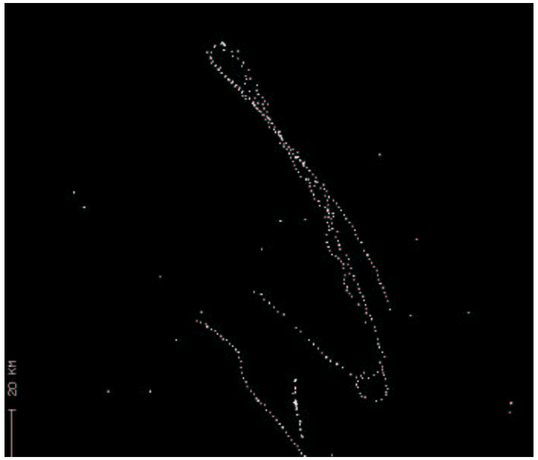
- E_k^{00}: Objects neither resolved, nor detected; all plots are false

$$\ell(Z_k, n_k, E_k^{00}|\mathbf{x}_k) = \text{const.} \times \mathcal{N}(\mathbf{O};\, \mathbf{H}(\mathbf{x}_k^1 - \mathbf{x}_k^2),\, \mathbf{R}_u)$$

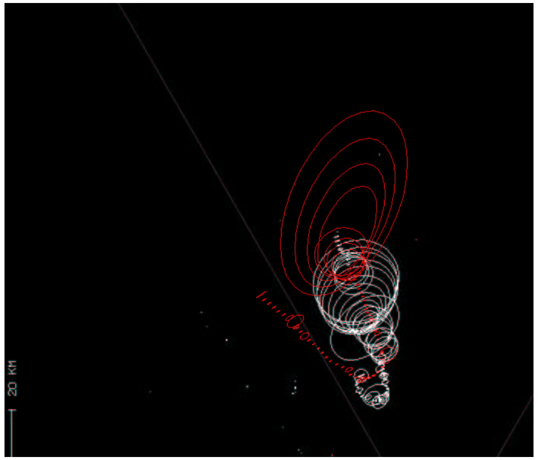
- E_k^{ij}: Objects resolved and individually detected, $\mathbf{z}_k^i, \mathbf{z}_k^j$ being the plots

$$\ell(Z_k, n_k|E_k^{ij}, \mathbf{x}_k) = \text{const.} \times [1 - P_u(\mathbf{x}_k)]\, \mathcal{N}\left(\begin{pmatrix}\mathbf{z}_k^i\\ \mathbf{z}_k^j\end{pmatrix}; \begin{pmatrix}\mathbf{H}_k\\ \mathbf{H}_k\end{pmatrix}\mathbf{x}_k, \begin{pmatrix}\mathbf{R}_k & \mathbf{O}\\ \mathbf{O} & \mathbf{R}_k\end{pmatrix}\right)$$

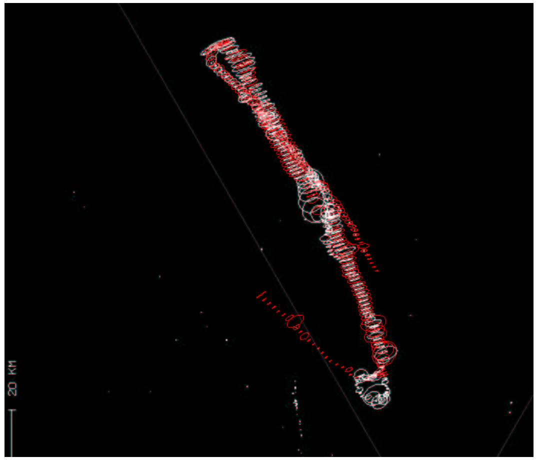
Mixtures with *negative* coefficients occur! Interpretation: Resolved targets keep a minimum distance, otherwise they were irresolvable.



radar raw data



no resolution model



with resolution model

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Tracking of Extended Targets and Target Groups

1. What is and what provides ‘tracking’?
2. Random matrices and object shape
3. Generalized track update (BAYES)
4. What are the perspectives?

Elements for Situation Pictures: Tracks of Time-varying Objects

Which object properties are of interest? $\rightarrow$ ***state*** X_k ***at time*** t_k

- road-moving vehicle: odometer count x_k $\quad X_k = (x_k, \dot{x}_k)$
- position, speed, acceleration: $\quad X_k = (\mathbf{r}_k, \dot{\mathbf{r}}_k, \ddot{\mathbf{r}}_k)$
- joint state of several objects: $\quad X_k = (\mathbf{x}_k^1, \mathbf{x}_k^2, \dots)$
- attributes, e.g. radar cross section $x_k \in \mathbb{R}^+$: $X_k = (\mathbf{x}_k, x_k)$
- maneuvering phase, object class $i_k \in \mathbb{N}$: $\quad X_k = (\mathbf{x}_k, i_k)$

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Elements for situation pictures: tracks of temporally evolving objects

Which object properties are of interest? → *state* X_k *at time* t_k

How to lean states X_k? → from *sensor data* $Z^k = \{Z_k, Z^{k-1}\}$, *context*

How to imprecise information? → *e.g. by conditional pdfs* $p(X_k|Z^k)$

What means “learning” in this context? → *iterative calculation of* $p(X_k|Z^k)$

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The general tracking equations

Prediction

$$p(X_k|Z^{k-1}) = \int dX_{k-1} \underbrace{p(X_k|X_{k-1})}_{\text{evolution}} \underbrace{p(X_{k-1}|Z^{k-1})}_{\text{filtering } t_{k-1}}$$

filtering

$$p(X_k|Z^k) = \frac{p(Z_k|X_k)\, p(X_k|Z^{k-1})}{\int dX_k \underbrace{p(Z_k|X_k)}_{\text{sensor model}} \underbrace{p(X_k|Z^{k-1})}_{\text{prediction}}}$$

retrodiction

$$p(X_l|Z^k) = \int dX_{l+1} \frac{\overbrace{p(X_{l+1}|X_l)}^{\text{evolution}} \overbrace{p(X_l|Z^l)}^{\text{filtering } t_l}}{\underbrace{p(X_{l+1}|Z^l)}_{\text{prediction } t_{l+1}}} \underbrace{p(X_{l+1}|Z^k)}_{\text{retrodiction } t_{l+1}}$$

Elements for situation pictures: tracks of temporally evolving objects

Which object properties are of interest? → *state* X_k *at time* t_k

How to lean states X_k? → from *sensor data* $Z^k = \{Z_k, Z^{k-1}\}$, *context*

How to imprecise information? → *e.g. by conditional pdfs* $p(X_k|Z^k)$

What means “learning” in this context? → *iterative calculation of* $p(X_k|Z^k)$

What is needed for this? → *evolution / sensor models* $p(X_k|X_{k-1}),\ p(Z_k|X_k)$

How to initiate / terminate tracking processes? → *sequential decisions*

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Why is 'tracking' a key function?

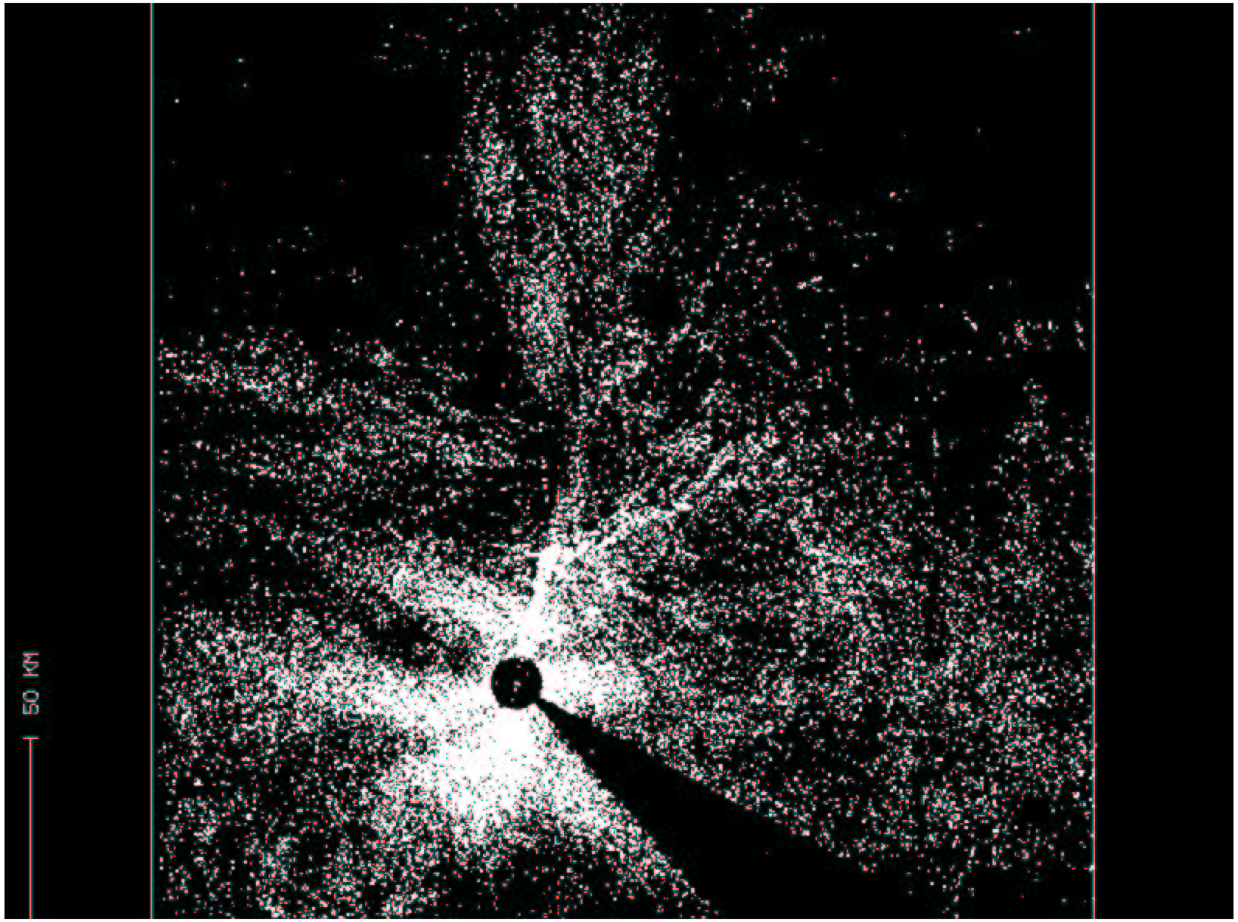
infer secondary quantities from incomplete measurement data.

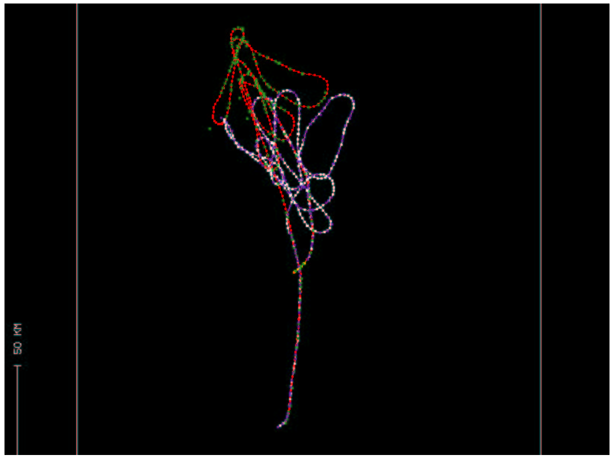
Eliminate fluctuating false return background (clutter).

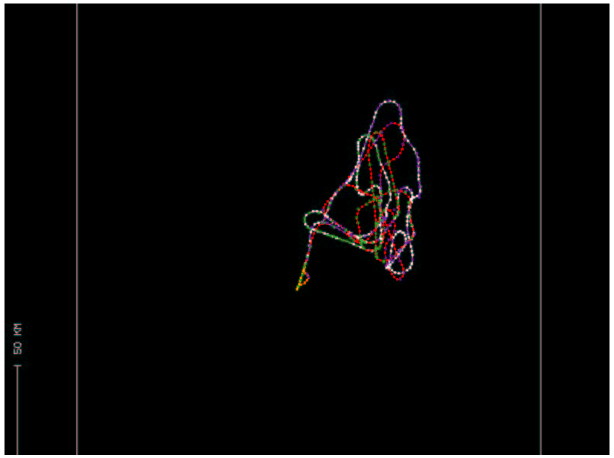
Create a time basis for classification from attribute data.

. . .

'Shape' of objects / object groups relevant in many applications.







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Track-based inference of object properties

- **Velocity history:** vehicle, helicopter, plane
- **Acceleration history:** threat: no under-wing weapons
- **Rare events:** truck by night on dirt road near a border
- **Object interrelations:** resulting from formation, convoy
- **Object sources / sinks:** classification by origin / designation
- **Classification:** road-moving vehicle, 'on-road' $\rightarrow$ 'off-road'

modeling: sensor data produced by extended objects

- actual measurement errors of individual scattering centers unimportant
- the ‘message’ of individual plots is dominated by the object extension
- individual plots to be interpreted as measurements of the object center
- related ‘measurement error’ proportional to extension *to be estimated*

object extension: ‘covariance-type’ matrices

- state augmentation by a random matrix
  - object extension exceeding resolution
  - closely spaced vehicle convoys
  - collectively moving object clouds

integration of available context information

- GMTI radar: Doppler blindness (MDV)
- digital road-map information (GIS)

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- maneuvering phase, object class $i_k \in \mathbb{N}$: $\quad X_k = (\mathbf{x}_k, i_k)$
- object shape: SPD random matrices $\mathbf{X}_k$ $\quad X_k = (\mathbf{x}_k, \mathbf{X}_k)$

extended objects: simplified description

- **kinematical state** at time t_k: x_k = (position, velocity, ...)
- **object extension** at time t_k: approximately by an ellipse
- **size:** volume, **shape:** ratio of semi-axes, spatial **orientation**
- extension: **SPD matrix** $\mathbf{X}_k$ (Symmetric, Positively Definite)

augmented state: kinematical state *vector* x_k, extension *matrix* $\mathbf{X}_k$

Generalize BAYESian tracking to extended objects.

n_k plots $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$ at time t_k, accumulated data $\mathcal{Z}^k = \{Z_k, n_k, \mathcal{Z}^{k-1}\}$

The conditional pdf $p(\mathrm{x}_k, \mathrm{X}_k | \mathcal{Z}^k)$ describes what is known about the extended object state $\mathrm{x}_k, \mathrm{X}_k$ based on all sensor data up to time t_k.

'extended object tracking': *iterative* calculation of $p(\mathrm{x}_k, \mathrm{X}_k | \mathcal{Z}^k)$.

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- $p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^k) = p(\mathbf{x}_k|\mathbf{X}_k, \mathcal{Z}^k)\; p(\mathbf{X}_k|\mathcal{Z}^k)$ **extended object 'track'**
- $p(Z_k, n_k|\mathbf{x}_k, \mathbf{X}_k)$ **sensor output to be processed, i.e. the likelihood**
- $p(\mathbf{x}_k|\mathcal{Z}^k) = \int d\mathbf{X}_k\; p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^k)$ **kinematics of the extended object**

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Generalize BAYESian tracking to extended objects.

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- $p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^k) = p(\mathbf{x}_k|\mathbf{X}_k, \mathcal{Z}^k)\, p(\mathbf{X}_k|\mathcal{Z}^k)$ **extended object 'track'**

  $p(\mathbf{x}_k|\mathbf{X}_k, \mathcal{Z}^k)$: **assume Gaussian density**

  $p(\mathbf{X}_k|\mathcal{Z}^k)$: **assume inverse Wishart density**

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modeling: sensor data produced by extended objects

- actual measurement errors of individual scattering centers unimportant
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- related 'measurement error' proportional to extension *to be estimated*

measurement eq.: $\mathbf{z}_k^j = (h_k^1 \mathbf{I}_d, h_k^2 \mathbf{I}_d, h_k^3 \mathbf{I}_d)\mathbf{x}_k + \mathbf{u}_k, \quad \mathbf{u}_k \sim \mathcal{N}(\mathbf{o}, \mathbf{R}_k)$

$$= (\mathbf{H}_k \otimes \mathbf{I}_d)\mathbf{x}_k + \mathbf{u}_k$$

simple example: $\mathrm{H}_k = (1, 0, 0)$ **(position measurement)**

measurement error: $\mathbf{R}_k \propto \mathbf{X}_k$ ***unknown!***

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Excursus: Properties of Kronecker products (1/3)

The Kronecker product $\mathrm{A} \otimes \mathrm{B}$ **of two matrices**

$\mathrm{A} = (a_{ij})_{i=1,j=1}^{m,n}$, B **is defined by:**

$$\mathbf{A} \otimes \mathrm{B} = \begin{pmatrix} a_{11}\mathrm{B} & a_{12}\mathrm{B} & \cdots & a_{1n}\mathrm{B} \\ a_{21}\mathrm{B} & a_{22}\mathrm{B} & \cdots & a_{2n}\mathrm{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathrm{B} & a_{m2}\mathrm{B} & \cdots & a_{mn}\mathrm{B} \end{pmatrix}.$$

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Excursus: Properties of Kronecker products (2/3)

For matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and a scalar α:

$$
\begin{aligned}
(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} &= \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) \\
\alpha \otimes \mathbf{A} &= \alpha\mathbf{A} = \mathbf{A}\alpha = \mathbf{A} \otimes \alpha \\
(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) &= \mathbf{AC} \otimes \mathbf{BD} \\
(\mathbf{A} \otimes \mathbf{B})^\top &= \mathbf{A}^\top \otimes \mathbf{B}^\top \\
(\mathbf{A} \otimes \mathbf{B})^{-1} &= \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}.
\end{aligned}
$$

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Excursus: Properties of Kronecker products (3/3)

For quadratic matrices A, B we obtain:

$$\mathrm{tr}[\mathbf{A} \otimes \mathbf{B}] = (\mathrm{tr}\mathbf{A})\,(\mathrm{tr}\mathbf{B}).$$

The determinant of $\mathrm{A} \otimes \mathrm{B}$ is given by the determinants of A, B

with $m = \mathrm{dim}(\mathbf{A})$, $n = \mathrm{dim}(\mathbf{B})$:

$$|\mathbf{A} \otimes \mathbf{B}| = |\mathbf{A}|^n |\mathbf{B}|^m.$$

modeling: 'collective' kinematics of extended objects

kinematics: $\mathbf{x}_k = \left(\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top\right)^\top, \quad d = \mathrm{dim}(\mathbf{r}_k),\ \mathrm{dim}(\mathbf{X}_k) = d \times d$

temporal evolution: $\mathbf{x}_k = \mathbf{\Phi}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim \mathcal{N}(\mathbf{o}, \mathbf{\Delta}_{k|k-1})$

evolution matrix: $\Phi_{k|k-1} = \begin{pmatrix} \mathbf{I}_d & \Delta t_k \mathbf{I}_d & \frac{1}{2}\Delta t_k^2 \mathbf{I}_d \\ \mathbf{O}_d & \mathbf{I}_d & \Delta t_k \mathbf{I}_d \\ \mathbf{O}_d & \mathbf{O}_d & \mathrm{e}^{-\Delta t_k/\theta}\mathbf{I}_d \end{pmatrix} = \mathbf{F}_{k|k-1} \otimes \mathbf{I}_d$

plant noise: $\boxed{\mathbf{\Delta}_{k|k-1} = \mathbf{D}_{k|k-1} \otimes \mathbf{X}_k}$ **important assumption!**

plant noise covariance $\mathrm{D}_{k|k-1} \otimes \mathrm{X}_k$: discussion of its structure

- ***formal argument:***
  - assuming this structure, there exists a mutually *conjugate pair* of pdfs $p(\mathrm{x}_k, \mathrm{X}_k|\mathcal{Z}^{k-1})$ and $p(Z_k, n_k|\mathrm{x}_k, \mathrm{X}_k)$; i.e. $p(\mathrm{x}_k, \mathrm{X}_k|\mathcal{Z}^k)$ belongs to the same family as $p(\mathrm{x}_k, \mathrm{X}_k|\mathcal{Z}^{k-1})$ $\Rightarrow$ analytical update equations!

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- ***practical arguments:***
  - Trajectories of group targets are the predictable, i.e. the inertial, the smaller their extension: 'maneuvering becomes dangerous'.
  - The larger a formation/convoy, the more probable are 'split-off' maneuvers; 'large' prediction covariances take this into account.
  - Large target groups/object swarms produce so many measurements that the prediction step becomes unimportant compared to filtering.

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prediction: kinematical state, object extension matrix

$$p(\mathbf{x}_k, \mathbf{X}_k | \mathcal{Z}^{k-1}) = \underbrace{p(\mathbf{x}_k | \mathbf{X}_k, \mathcal{Z}^{k-1})}_{\textit{vector}\ \text{variate}} \ \underbrace{p(\mathbf{X}_k | \mathcal{Z}^{k-1})}_{\textit{matrix}\ \text{variate}}$$

track structure at time t_{k-1} $\quad p(\mathbf{x}_{k-1} | \mathbf{X}_k, \mathcal{Z}^{k-1}) = \mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}_{k-1|k-1},\ \mathbf{P}_{k-1|k-1} \otimes \mathbf{X}_k\big)$

is preserved after prediction to t_k: $\quad p(\mathbf{x}_k | \mathbf{X}_k, \mathcal{Z}^{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1} \otimes \mathbf{X}_k\big)$

with: $\mathbf{x}_{k|k-1} = (\mathbf{F}_{k|k-1} \otimes \mathbf{I}_d)\mathbf{x}_{k-1|k-1}, \quad \mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}^{\top} + \mathbf{D}_{k|k-1}$ **KALMAN type prediction**

prediction: kinematical state, object extension matrix

$$p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^{k-1}) = \underbrace{p(\mathbf{x}_k|\mathbf{X}_k, \mathcal{Z}^{k-1})}_{\textbf{\textit{vector} variate}} \; \underbrace{p(\mathbf{X}_k|\mathcal{Z}^{k-1})}_{\textbf{\textit{matrix} variate}}$$

track structure at time t_{k-1} $\quad p(\mathbf{x}_{k-1}|\mathbf{X}_k, \mathcal{Z}^{k-1}) = \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1} \otimes \mathbf{X}_k\big)$

is preserved after prediction to t_k**:** $\quad p(\mathbf{x}_k|\mathbf{X}_k, \mathcal{Z}^{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1} \otimes \mathbf{X}_k\big)$

with: $\mathbf{x}_{k|k-1} = (\mathbf{F}_{k|k-1} \otimes \mathbf{I}_d)\mathbf{x}_{k-1|k-1}, \quad \mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\top}_{k|k-1} + \mathbf{D}_{k|k-1}$ **KALMAN type prediction**

Assume for $p(\mathbf{X}_k|\mathcal{Z}^{k-1})$ **and** $p(\mathbf{X}_k|\mathcal{Z}^k)$ ***inverse* WISHART densities:**

$$p(\mathbf{X}_k|\mathcal{Z}^{k-1}) = \mathcal{IW}\big(\mathbf{X}_k;\, \nu_{k|k-1}, \mathbf{X}_{k|k-1}\big) \propto |\mathbf{X}_k|^{-\frac{\nu_{k|k-1}+d+1}{2}} \; \mathrm{etr}\big[-\tfrac{1}{2}\mathbf{X}_{k|k-1}\mathbf{X}_k^{-1}\big].$$

$$\mathbb{E}[\mathbf{X}_k] = \frac{\mathbf{X}_{k|k-1}}{\nu_{k|k-1}-d-1}, \quad \mathrm{etr}[\mathbf{A}] = \exp[\mathrm{tr}\mathbf{A}], \; |\mathbf{A}| = \det \mathbf{A}$$

Calculate $\nu_{k|k-1}$, $\mathbf{X}_{k|k-1}$ **from** $\nu_{k-1|k-1}$, $\mathbf{X}_{k-1|k-1}$ **(i.e. the filtering)!**

structure of the measurement likelihood function

in general: $p(Z_k, n_k|\mathbf{x}_k, \mathbf{X}_k, \mathcal{Z}^{k-1}) = p(Z_k|n_k, \mathbf{x}_k, \mathbf{X}_k)\, p(n_k|\mathbf{x}_k, \mathbf{X}_k)$

at present: no false returns, unresolved measurements are allowed.

$$p(Z_k|n_k, \mathbf{x}_k, \mathbf{X}_k) = \prod_{j=1}^{n_k} \mathcal{N}\Big(\mathbf{z}_k^j;\ (\mathbf{H}_k \otimes \mathbf{I}_d)\mathbf{x}_k,\ \mathbf{X}_k\Big) \quad \text{(independent plots)}$$

$$\propto \mathcal{N}\Big(\mathbf{z}_k;\ (\mathbf{H}_k \otimes \mathbf{I}_d)\mathbf{x}_k,\ \tfrac{1}{n_k}\mathbf{X}_k\Big)\ \mathcal{LW}\Big(\mathbf{Z}_k;\ n_k - 1,\ \mathbf{X}_k\Big)$$

$$\mathbf{z}_k = \frac{1}{n_k}\sum_{j=1}^{n_k} \mathbf{z}_k^j, \quad \mathbf{Z}_k = \sum_{j=1}^{n_k} (\mathbf{z}_k^j - \mathbf{z}_k)(\mathbf{z}_k^j - \mathbf{z}_k)^\top$$

$$\mathcal{LW}\big(\mathbf{Z}_k;\ n_k - 1,\ \mathbf{X}_k\big) = |\mathbf{X}_k|^{-\frac{n_k-1}{2}}\ \mathrm{etr}\big[-\tfrac{1}{2}\mathbf{Z}_k\mathbf{X}_k^{-1}\big]$$

Excursus: some very useful formulae

For column vectors $\mathbf{x}$, $\mathbf{y}$ of equal dimension, the following identities are valid:

$$
\begin{aligned}
\mathbf{x}^\top \mathbf{y} &= \mathrm{tr}\Big[\mathbf{x}\mathbf{y}^\top\Big] \\
\mathbf{x}^\top \mathbf{A}^{-1}\mathbf{x} &= \mathrm{tr}\Big[\mathbf{x}\mathbf{x}^\top \mathbf{A}^{-1}\Big] = \mathrm{tr}\Big[\mathbf{A}^{-1}\mathbf{x}\mathbf{x}^\top\Big] \\
\exp\Big[-\tfrac{1}{2}\mathbf{x}^\top \mathbf{A}^{-1}\mathbf{x}\Big] &= \mathrm{etr}\Big[-\tfrac{1}{2}\mathbf{x}\mathbf{x}^\top \mathbf{A}^{-1}\Big] \\
\Big|\mathbf{I} + \mathbf{x}\mathbf{y}^\top\Big| &= 1 + \mathbf{x}^\top \mathbf{y}.
\end{aligned}
$$

$\mathrm{etr}[\mathbf{A}]$ is an abbreviation for $\exp[\mathrm{tr}\mathbf{A}]$. For proofs see:

Harville, D. A., *Matrix Algebra from a Statistician’s Perspective*. Springer, 1997.

Excursus: How find the Wishart density?

By applying the product formula for Gaussians repeatedly we obtain:

$$p(Z_k|n_k, \mathrm{x}_k, \mathbf{X}_k) = \mathcal{N}\Big(\mathbf{z}_k;\, \mathbf{H}_k\mathrm{x}_k,\, \tfrac{1}{n_k}\mathbf{X}_k\Big) \prod_{i=1}^{n_k-1} \mathcal{N}\Big(\mathbf{z}_k^{i+1};\, \bar{\mathbf{z}}_k^i,\, \tfrac{i+1}{i}\mathbf{X}_k\Big)$$

with: $\bar{\mathbf{z}}_k^i = \frac{1}{i}\sum_{j=1}^{i}\mathbf{z}_k^j, \quad \mathbf{z}_k = \bar{\mathbf{z}}_k^{n_k} = \frac{1}{n_k}\sum_{j=1}^{i}\mathbf{z}_k^j.$

Only the first factor of the right side of this equation depends on the kinematic state variable x_k. The remaining $n_k - 1$ factors are functions of the extension $\mathbf{X}_k$ alone. An induction argument yields:

$$\prod_{i=1}^{n_k-1} \mathcal{N}\Big(\mathbf{z}_k^{i+1};\, \bar{\mathbf{z}}_k^i,\, \tfrac{i+1}{i}\mathbf{X}_k\Big) \propto |\mathbf{X}_k|^{-\frac{n_k-1}{2}}\, \mathrm{etr}\Big[-\tfrac{1}{2}\mathbf{Z}_k\mathbf{X}_k^{-1}\Big]$$

$$\propto \mathcal{LW}\Big(\mathbf{Z}_k;\, n_k - 1,\, \mathbf{X}_k\Big).$$

structure of joint state pdf after the filtering step

exploiting BAYES: $$p(Z_k|n_k, \mathbf{x}_k, \mathbf{X}_k)\, p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^{k-1}) \propto \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k} \otimes \mathbf{X}_k\big)}_{\text{KALMAN type update}}$$

$$\times \underbrace{|\mathbf{X}_k|^{-\frac{1}{2}}\ \mathrm{etr}\big[-\tfrac{1}{2}\mathbf{N}_{k|k-1}\mathbf{X}_k^{-1}\big]}_{\text{innovation factor}}\ \underbrace{\mathcal{LW}\big(\mathbf{Z}_k;\, n_k-1,\, \mathbf{X}_k\big)}_{\text{from measurement likelihood}}\ \underbrace{\mathcal{IW}\big(\mathbf{X}_k;\, \nu_{k|k-1},\, \mathbf{X}_{k|k-1}\big)}_{\text{extension prediction}}$$

up to a factor independent of $\mathbf{x}_k$, $\mathbf{X}_k$ with *innovation matrix* $\mathbf{N}_{k|k-1}$ and innov. cov. $S_{k|k-1}$:

$$\mathbf{N}_{k|k-1} = S_{k|k-1}^{-1}\big(\mathbf{z}_k - (\mathbf{H}_k \otimes \mathbf{I}_d)\mathbf{x}_{k|k-1}\big)\big(\mathbf{z}_k - (\mathbf{H}_k \otimes \mathbf{I}_d)\mathbf{x}_{k|k-1}\big)^\top$$

Fraunhofer FKIE

structure of joint state pdf after the filtering step

exploiting BAYES: $$p(Z_k|n_k, \mathbf{x}_k, \mathbf{X}_k)\, p(\mathbf{x}_k, \mathbf{X}_k|\mathbf{Z}^{k-1}) \propto \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k} \otimes \mathbf{X}_k\big)}_{\text{KALMAN type update}}$$

$$\times \underbrace{|\mathbf{X}_k|^{-\frac{1}{2}}\ \mathrm{etr}\big[-\tfrac{1}{2}\mathbf{N}_{k|k-1}\mathbf{X}_k^{-1}\big]}_{\text{innovation factor}}\ \underbrace{\mathcal{LW}\big(\mathbf{Z}_k;\, n_k - 1,\, \mathbf{X}_k\big)}_{\text{from measurement likelihood}}\ \underbrace{\mathcal{IW}\big(\mathbf{X}_k;\, \nu_{k|k-1},\, \mathbf{X}_{k|k-1}\big)}_{\text{extension prediction}}$$

up to a factor independent of $\mathbf{x}_k$, $\mathbf{X}_k$ with *innovation matrix* $\mathbf{N}_{k|k-1}$ and innov. cov. $S_{k|k-1}$:

$$\mathbf{N}_{k|k-1} = S_{k|k-1}^{-1}\big(\mathbf{z}_k - (\mathbf{H}_k \otimes \mathbf{I}_d)\mathbf{x}_{k|k-1}\big)\big(\mathbf{z}_k - (\mathbf{H}_k \otimes \mathbf{I}_d)\mathbf{x}_{k|k-1}\big)^\top$$

finally: $$p(Z_k|n_k, \mathbf{x}_k, \mathbf{X}_k)\, p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^{k-1}) \propto \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k} \otimes \mathbf{X}_k\big)}_{\text{KALMAN type update}}\ \underbrace{\mathcal{IW}\big(\mathbf{X}_k;\, \nu_{k|k},\, \mathbf{X}_{k|k}\big)}_{\text{extension update}}$$

with simple update equations: $$\mathbf{X}_{k|k} = \mathbf{X}_{k|k-1} + \mathbf{N}_{k|k-1} + \mathbf{Z}_k, \quad \nu_{k|k} = \nu_{k|k-1} + n_k.$$

remark: due to the innovation matrix $\mathbf{N}_{k|k-1}$ also in case of point targets the estimation of an unknown measurement error is possible or the ‘extension’ of an unresolved target group.

Fraunhofer FKIE

marginalization of the joint density $p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^k)$

sometimes: interest in estimates of the kinematical state $\mathbf{x}_k$ only

$$
\begin{aligned}
p(\mathbf{x}_k|\mathcal{Z}_k) &= \int d\mathbf{X}_k\, p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^k) \\
&= \int d\mathbf{X}_k\; \mathcal{N}\Big(\mathbf{x}_k;\; \mathbf{x}_{k|k},\; \mathbf{P}_{k|k} \otimes \mathbf{X}_k\Big)\; \mathcal{IW}\Big(\mathbf{X}_k;\; \nu_{k|k},\; \mathbf{X}_{k|k}\Big)
\end{aligned}
$$

Fraunhofer FKIE

marginalization of the joint density $p(\mathrm{x}_k, \mathrm{X}_k|\mathcal{Z}^k)$

sometimes: interest in estimates of the kinematical state x_k only

$$
\begin{aligned}
p(\mathbf{x}_k|\mathcal{Z}_k) &= \int d\mathbf{X}_k \, p(\mathbf{x}_k, \mathbf{X}_k|\mathcal{Z}^k) \\
&= \int d\mathbf{X}_k \; \mathcal{N}\Big(\mathbf{x}_k;\; \mathbf{x}_{k|k},\; \mathbf{P}_{k|k} \otimes \mathbf{X}_k\Big) \; \mathcal{IW}\Big(\mathbf{X}_k;\; \nu_{k|k},\; \mathbf{X}_{k|k}\Big) \\
&= \mathcal{T}\Big(\mathbf{x}_k;\; \nu_{k|k} + s(1-d),\; \mathbf{x}_{k|k}, \mathbf{P}_{k|k} \otimes \mathbf{X}_{k|k}\Big) \quad \textbf{(standard algebra!)}
\end{aligned}
$$

multivariate Student-t-density with $\nu_{k|k}$ degrees of freedom

basis for kinematical state estimates with related error covariance matrices,

calculation of expectation gates, extended object track-to-track correlation/fusion, ...