Likelihood Functions

The likelihood function answers the question:

What does the sensor tell about the state x of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1$, $\rho_F = 0$

  at each time one measurement: $\boxed{p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k; \mathbf{H}\mathbf{x}_k, \mathbf{R})}$

- **real conditions, one object:** $P_D < 1$, $\rho_F > 0$

  at each time n_k measurements $Z_k = \{\mathbf{z}_k^1, \dots, \mathbf{z}_k^{n_k}\}$!

$$\boxed{p(Z_k, n_k|\mathbf{x}_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\ \mathbf{H}\mathbf{x}_k,\ \mathbf{R}\big)}$$

Bayes Filtering for: $P_D < 1, \rho_F > 0$, well-separated objects

state $\mathbf{x}_k$, ***current* data** $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{m_k}$, ***accumulated* data** $\mathcal{Z}^k = \{Z_k, \mathcal{Z}^{k-1}\}$

interpretation hypotheses E_k for Z_k

$$\left.\begin{array}{l}\text{object not detected, } 1 - P_D \\ \mathbf{z}_k \in Z_k \text{ from object, } P_D\end{array}\right\} \quad m_k + 1 \text{ interpretations}$$

interpretation histories H_k for $\mathcal{Z}^k$

- tree structure: $H_k = (E_{H_k}, H_{k-1}) \in \mathcal{H}^k$
- current: E_{H_k}, *pre*histories: H_{k-i}

$$p(\mathbf{x}_k|\,\mathcal{Z}^k) = \sum_{H_k} p(\mathbf{x}_k, H_k|\,\mathcal{Z}^k) = \sum_{H_k} \underbrace{p(H_k|\,\mathcal{Z}^k)}_{\text{weight!}}\ \underbrace{p(\mathbf{x}_k|\,H_k, \mathcal{Z}^k)}_{\textit{given } H_k\text{: unique}}$$

'mixture' density

Closer look: $P_D < 1, \rho_F > 0$, well-separated targets

filtering (at time t_{k-1}): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{H_{k-1}} p_{H_{k-1}} \, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{H_{k-1}},\, \mathbf{P}_{H_{k-1}}\big)$

prediction (for time t_k):

$$
\begin{aligned}
p(\mathbf{x}_k|\mathcal{Z}^{k-1}) &= \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) && \text{(MARKOV model)}\\
&= \sum_{H_{k-1}} p_{H_{k-1}}\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}\mathbf{x}_{H_{k-1}},\, \mathbf{F}\mathbf{P}_{H_{k-1}}\mathbf{F}^\top + \mathbf{D}\big)
\end{aligned}
$$

measurement likelihood:

$$
\begin{aligned}
p(Z_k, m_k|\mathbf{x}_k) &= \sum_{j=0}^{m_k} p(\mathcal{Z}_k|E_k^j, \mathbf{x}_k, m_k)\, P(E_k^j|x_k, m_k) && (E_k^j\text{: interpretations})\\
&\propto (1 - P_D)\,\rho_F + P_D \sum_{j=1}^{m_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big) && (\mathbf{H}, \mathbf{R}, P_D, \rho_F)
\end{aligned}
$$

filtering (at time t_k):

$$
\begin{aligned}
p(\mathbf{x}_k|\mathcal{Z}^k) &\propto p(Z_k, m_k|\mathbf{x}_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1}) && \text{(BAYES' rule)}\\
&= \sum_{H_k} p_{H_k}\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{H_k},\, \mathbf{P}_{H_k}\big) && \text{(Exploit product formula)}
\end{aligned}
$$

Fraunhofer FKIE

Exercise: **Show that $p(\mathbf{x}_k|\mathcal{Z}^k)$ is given by Kalman updates and weights $p^j_{H_k}$.**

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \sum_{j=0}^{m_k} \sum_{H_{k-1}} p^j_{H_{k-1}} \, \mathcal{N}\Big(\mathbf{x}_{k-1};\, \mathbf{x}^j_{H_{k-1}},\, \mathbf{P}^j_{H_{k-1}}\Big)$$

$$p^j_{H_{k-1}} = \frac{p^{j*}_{H_{k-1}}}{\sum_j p^{j*}_{H_{k-1}}}$$

$$p^{j*}_{H_{k-1}} = p_{H_{k-1}} \begin{cases} (1-P_D)\,\rho_F & j=0 \\ \dfrac{P_D}{\sqrt{|2\pi \mathbf{S}^j_{H_{k-1}}|}}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{\nu}^{j\top}_{H_k} \mathbf{S}^{j\,-1}_{H_{k-1}} \boldsymbol{\nu}^j_{H_k}} & j\neq 0 \end{cases}$$

Insert mixtures and exploit product formula in the numerator and denominator!

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m data, N hypotheses $\rightarrow N^{m+1}$ continuations

radical solution: mono-hypothesis approximation

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  $\rightarrow$ KALMAN filter (KF)

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- Force a **unique interpretation** in case of a conflict!

  look for *smallest statistical distance:* $\min_i ||\boldsymbol{\nu}^i_{k|k-1}||$

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  + one hypothesis, – hard decision, – not adaptive

- **global combining:** Merge all hypotheses!

  $\to$ PDAF, JPDAF filter

  + all data, + adaptive, – reduced applicability

PDAF Filter: formally analog to Kalman Filter

Filtering (scan $k-1$): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})$ (→ initiation)

prediction (scan k): $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$ (like Kalman)

Filtering (scan k): $p(\mathbf{x}_k|\mathcal{Z}^k) \approx \sum_{j=0}^{m_k} p_k^j \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}^j, \mathbf{P}_{k|k}^j) \approx \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

$$\boldsymbol{\nu}_k = \boxed{\textstyle\sum_{j=0}^{m_k} p_k^j\, \boldsymbol{\nu}_k^j}, \qquad \boldsymbol{\nu}_k^j = \mathbf{z}_k^j - \mathbf{H}\mathbf{x}_{k|k-1}$$ *combined* innovation

$$\mathbf{W}_k = \mathbf{P}_{k|k-1}\mathbf{H}^\top\mathbf{S}_k^{-1}, \qquad \mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}_k$$ Kalman gain matrix

$$p_k^j = p_k^{j*} \,/\, \textstyle\sum_j p_k^{j*}, \qquad p_k^{j*} = \begin{cases} (1-P_D)\,\rho_F \\ \frac{P_D}{\sqrt{|2\pi\mathbf{S}_{H_k}|}}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{\nu}_{H_k}^\top \mathbf{S}_{H_k}\boldsymbol{\nu}_{H_k}} \end{cases}$$ weighting factors

$$\mathbf{x}_k = \mathbf{x}_{k|k-1} + \mathbf{W}_k\,\boldsymbol{\nu}_k$$ (Filtering Update: Kalman)

$$\mathbf{P}_k = \mathbf{P}_{k|k-1} - (1-p_k^0)\,\mathbf{W}_k\mathbf{S}\mathbf{W}_k^\top$$ (Kalman part)

$$+\, \mathbf{W}_k\Big\{\textstyle\sum_{j=0}^{m_k} p_k^j\, \boldsymbol{\nu}_k^j\boldsymbol{\nu}_k^{j\top} - \boldsymbol{\nu}_k\boldsymbol{\nu}_k^\top\Big\}\mathbf{W}_k^\top$$ (Spread of Innovations)

The qualitative shape of $p(\mathbf{x}_k|\mathcal{Z}^k)$ is often much simpler than its correct representation: *a few pronounced modes*

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  after calculating the weights p_{H_k}, before filtering

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Fraunhofer FKIE

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- **local combining:** Merge *similar hypotheses*!

  after the complete calculation of the pdfs

  $\rightarrow$ *limiting case:* PDAF (global combining)

Successive Local Combining

Partial sums of *similar* densities $\rightarrow$ moment matching:

$$\boxed{\sum_{H_k \in \mathcal{H}^{k*}} p_{H_k} \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{H_k}, \mathbf{P}_{H_k}) \approx p_{H_k^*} \mathcal{N}(\mathbf{x}_k; \mathbf{x}_{H_k^*}, \mathbf{P}_{H_k^*})}$$

$\mathcal{H}^{k*} \subset \mathcal{H}^k \quad \rightarrow \quad H_k^*$: *effective* hypothesis

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similarity: $\quad d(H_1, H_2) < \mu \quad$ mit (z.B.):

$$d(H_1, H_2) = (\mathbf{x}_{H_1} - \mathbf{x}_{H_2})^\top (\mathbf{P}_{H_1} + \mathbf{P}_{H_2})^{-1} (\mathbf{x}_{H_1} - \mathbf{x}_{H_2})$$

Start: Hypothesis of highest weight $H_1 \rightarrow$ search similar hypothesis ($p_H \searrow$) $\rightarrow$ merge: $(H_1, H) \succ H_1^* \rightarrow$ continue search ($p_H \searrow$) ...

$\rightarrow$ **restart:** hypothesis with next to highest weight $H_2 \rightarrow \dots$

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- In many cases: good approximations $\to$ *quasi-optimality*
- PDAF, JPDAF: $\mathcal{H}^{k*} = \mathcal{H}^k \to$ *limited applicability*
- robustness $\to$ *detail mostly irrelevant*

Retrodiction for GAUSSian Mixtures

$$\boxed{\textbf{wanted:} \quad p(\mathbf{x}_l|\mathcal{Z}^k) \longleftarrow p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \quad \text{for} \quad l < k}$$

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{H_k} p(\mathbf{x}_l, H_k|\mathcal{Z}_k) = \sum_{H_k} \underbrace{p(\mathbf{x}_l|H_k, \mathcal{Z}^k)}_{\text{no ambiguities!}} \underbrace{p(H_k|\mathcal{Z}^k)}_{\text{filtering!}}$$

Calculation of $p(\mathbf{x}_l|H_k, \mathcal{Z}^k)$ as in case of $P_D = 1$, $\rho_F = 0$!

$$p(\mathbf{x}_l|H_k, \mathcal{Z}^k) = \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}_{H_k}(l|k),\, \mathbf{P}_{H_k}(l|k)\Big)$$

with parameters given by RAUCH-TUNG-STRIEBEL formulae:

$$\mathbf{x}_{H_k}(l|k) = \mathbf{x}_{H_k}(l|l) + \mathbf{W}_{H_k}(l|k)\,\big(\mathbf{x}_{H_k}(l+1|k) - \mathbf{x}_{H_k}(l+1|l)\big)$$
$$\mathbf{P}_{H_k}(l|k) = \mathbf{P}_{H_k}(l|l) + \mathbf{W}_{H_k}(l|k)\,\big(\mathbf{P}_{H_k}(l+1|k) - \mathbf{P}_{H_k}(l+1|l)\big)\,\mathbf{W}_{H_k}(l|k)^\top$$

gain matrix: $\mathbf{W}_{H_k}(l|k) = \mathbf{P}_{H_k}(l|l)\mathbf{F}_{l+1|l}^\top \mathbf{P}_{H_k}(l+1|l)^{-1}$

Retrodiction of Hypotheses' Weights

Consider approximation: neglect RTS step!

$$p(\mathbf{x}_l|H_k, \mathcal{Z}^k) = \mathcal{N}\big(\mathbf{x}_l;\ \mathbf{x}_{H_k}(l|k),\ \mathbf{P}_{H_k}(l|k)\big) \approx \mathcal{N}\big(\mathbf{x}_l;\ \mathbf{x}_{H_k}(l|l),\ \mathbf{P}_{H_k}(l|l)\big)$$

$$p(\mathbf{x}_l|H_k, \mathcal{Z}^k) \approx \sum_{H_l} p^*_{H_l}\ \mathcal{N}\big(\mathbf{x}_l;\ \mathbf{x}_{H_k}(l|l),\ \mathbf{P}_{H_k}(l|l)\big)$$

with recursively defined weights: $\boxed{p^*_{H_k} = p_{H_k}, \quad p^*_{H_l} = \sum p^*_{H_{l+1}}}$

summation over all histories H_{l+1} with equal pre-histories!

- Strong sons strengthen weak fathers.
- Weak sons weaken even strong fathers.
- If all sons die, also the father must die.

Track Extraction: Initiation of the PDF Iteration

***extraction of target tracks*: detection on a higher level of abstraction**

start: data sets $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{m_k}$ (sensor performance: P_D, ρ_F, $\mathbf{R}$)

goal: Detect a target trajectory in a time series: $\mathcal{Z}^k = \{Z_i\}_{i=1}^k$!

at first simplifying assumptions:

- The targets in the sensors' field of view (FoV) are well-separated.
- The sensor data in the FoV in scan i are produced simultaneously.

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decision between two competing hypotheses:

h_1: Besides false returns $\mathcal{Z}^k$ contains also target measurements.

h_0: There is no target existing in the FoV; all data in $\mathcal{Z}^k$ are false.

statistical decision errors:

$P_1 = \text{Prob}(\textit{accept}\, h_1 | h_1)$ analogous to the sensors' P_D

$P_0 = \text{Prob}(\textit{accept}\, h_1 | h_0)$ analogous to the sensors' P_F

Fraunhofer FKIE

Practical Approach: Sequential Likelihood Ratio Test

Goal: Decide as fast as possible for given decision errors P_0, P_1!

Consider the ratio of the conditional probabilities $p(h_1|\mathcal{Z}^k)$, $p(h_0|\mathcal{Z}^k)$ and the likelihood ratio $\mathrm{LR}(k) = p(\mathcal{Z}^k|h_1)/p(\mathcal{Z}^k|h_0)$ as an intuitive decision function:

$$\frac{p(h_1|\mathcal{Z}^k)}{p(h_0|\mathcal{Z}^k)} = \frac{p(\mathcal{Z}^k|h_1)}{p(\mathcal{Z}^k|h_0)}\,\frac{p(h_1)}{p(h_0)} \qquad \text{a priori: } p(h_1) = p(h_0)$$

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Starting from a time window with length $k = 1$, calculate the test function $\mathrm{LR}(k)$ successively and compare it with *two* thresholds A, B:

If	$\mathrm{LR}(k) < A$,	accept hypothesis h_0 (i.e. no target is existing)!
If	$\mathrm{LR}(k) > B$,	accept hypothesis h_1 (i.e. target exists in FoV)!
If	$A < \mathrm{LR}(k) < B$,	wait for new data Z_{k+1}, repeat with $\mathrm{LR}(k+1)$!

Sequential LR Test: Some Useful Properties

1. Thresholds and decision errors are approximately related to each other by:

$$\boxed{A \approx \frac{1 - P_1}{1 - P_0} \quad \text{and} \quad B \approx \frac{P_1}{P_0}}$$

2. The *actual* decision length (number of scans required) is a random variable.

3. On average, the test has a *minimal* decision length for given errors P_0, P_1.

4. The quantity P_0 (P_1) affects the *mean* decision length given h_1 (h_0) holds.

5. Choose the probability P_1 close to 1 for actually detecting real target tracks.

6. P_0 should be small for not overloading the tracking system with false tracks.

Iterative Calculation of the Likelihood Ratio

$$\mathsf{LR}(k) = \frac{p(\mathcal{Z}^k|h_1)}{p(\mathcal{Z}^k|h_0)} = \frac{\int d\mathbf{x}_k \, p(Z_k, m_k, \mathbf{x}_k, \mathcal{Z}^{k-1}|h_1)}{p(Z_k, m_k, \mathcal{Z}^{k-1}, h_0)}$$

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basic idea: iterative calculation!

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$$= \frac{\int d\mathbf{x}_k\, p(Z_k, m_k|\mathbf{x}_k, h_1)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1}, h_1)}{|\mathsf{FoV}|^{-m_k}\, p_F(m_k)}\, \mathsf{LR}(k-1)$$

basic idea: iterative calculation!

Let $H_k = \{E_k, H_{k-1}\}$ be an interpretation history of the time series $\mathcal{Z}^k = \{Z_k, \mathcal{Z}^{k-1}\}$.

$E_k = E_k^0$: target was not detected, $E_k = E_k^j$: $\mathbf{z}_k^j \in Z_k$ is a target measurement.

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}, h_1) = \sum_{H_{k-1}} p(\mathbf{x}_k|H_{k-1}\mathcal{Z}^{k-1}, h_1)\, p(H_{k-1}|\mathcal{Z}^{k-1}, h_1)$$ The standard MHT prediction!

$$p(Z_k, m_k|\mathbf{x}_k, h_1, h_1) = \sum_{E_k} p(Z_k, E_k|\mathbf{x}_k, h_1)$$ The standard MHT likelihood function!

The calculation of the likelihood ratio is just *a by-product of Bayesian MHT tracking*.

Iteration Formula for $\mathsf{LR}(k) = p(\mathcal{Z}^k|h_1)/p(\mathcal{Z}^k|h_0)$

initiation: $k = 0, \quad \mathbf{j}_0 = 0, \quad \lambda_{\mathbf{j}_0} = 1$

recursion: $$\mathsf{LR}(k+1) = \sum_{\mathbf{j}_{k+1}} \lambda_{\mathbf{j}_{k+1}} = \sum_{j_{k+1}=0}^{m_{k+1}} \sum_{\mathbf{j}_k} \lambda_{j_{k+1}\mathbf{j}_k} \lambda_{\mathbf{j}_k}$$

with: $$\lambda_{j_{k+1}\mathbf{j}_k} = \begin{cases} 1 - P_D & \text{for} \quad j_{k+1} = 0 \\ \frac{P_D}{\rho_F} \mathcal{N}(\boldsymbol{\nu}_{j_{k+1}\mathbf{j}_k}, \mathbf{S}_{j_{k+1}\mathbf{j}_k}) & \text{for} \quad j_{k+1} \neq 0 \end{cases}$$

convenient notation: with $\mathbf{j}_k = (j_k, \ldots, j_1)$ let $\sum_{\mathbf{j}_k} \lambda_{\mathbf{j}_k} = \sum_{j_k=0}^{m_k} \cdots \sum_{j_1=0}^{m_1} \lambda_{j_k \ldots j_1}$

Fraunhofer FKIE

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with: $\lambda_{j_{k+1}\mathbf{j}_k} = \begin{cases} 1 - P_D & \text{for} \quad j_{k+1} = 0 \\ \frac{P_D}{\rho_F} \mathcal{N}(\boldsymbol{\nu}_{j_{k+1}\mathbf{j}_k}, \mathbf{S}_{j_{k+1}\mathbf{j}_k}) & \text{for} \quad j_{k+1} \neq 0 \end{cases}$

innovation: $\boldsymbol{\nu}_{j_{k+1}\mathbf{j}_k} = \mathbf{z}_{j_{k+1}} - \mathbf{H}_{j_{k+1}} \mathbf{x}_{\mathbf{j}_{k+1|k}}$

innov. cov.: $\mathbf{S}_{j_{k+1}\mathbf{j}_k} = \mathbf{H}_{j_{k+1}} \mathbf{P}_{\mathbf{j}_{k+1|k}} \mathbf{H}_{j_{k+1}}^\top + \mathbf{R}_{j_{k+1}}$

state update: $\mathbf{x}_{\mathbf{j}_{k+1|k}} = \mathbf{F}_{j_{k+1}} \mathbf{x}_{\mathbf{j}_k} \qquad \mathbf{x}_{\mathbf{j}_k} = \mathbf{x}_{\mathbf{j}_{k|k-1}} + \mathbf{W}_{j_k\mathbf{j}_{k-1}} \boldsymbol{\nu}_{j_k,\mathbf{j}_{k-1}}$

covariances: $\mathbf{P}_{\mathbf{j}_{k+1|k}} = \mathbf{F}_{j_{k+1}} \mathbf{P}_{\mathbf{j}_k} \mathbf{F}_{j_{k+1}}^\top + \mathbf{D}_{j_{k+1}} \qquad \mathbf{P}_{\mathbf{j}_k} = \mathbf{P}_{\mathbf{j}_{k|k-1}} - \mathbf{W}_{j_k\mathbf{j}_{k-1}} \mathbf{S}_{j_k\mathbf{j}_{k-1}} \mathbf{W}_{j_k\mathbf{j}_{k-1}}^\top$

Iteration Formula for $\mathsf{LR}(k) = p(\mathcal{Z}^k|h_1)/p(\mathcal{Z}^k|h_0)$

initiation: $k = 0, \quad \mathrm{j}_0 = 0, \quad \lambda_{\mathrm{j}_0} = 1$

recursion: $$\mathsf{LR}(k+1) = \sum_{\mathrm{j}_{k+1}} \lambda_{\mathrm{j}_{k+1}} = \sum_{j_{k+1}=0}^{m_{k+1}} \sum_{\mathrm{j}_k} \lambda_{j_{k+1}\mathrm{j}_k} \lambda_{\mathrm{j}_k}$$

with: $$\lambda_{j_{k+1}\mathrm{j}_k} = \begin{cases} 1 - P_D & \text{for} \quad j_{k+1} = 0 \\ \frac{P_D}{\rho_F} \mathcal{N}(\boldsymbol{\nu}_{j_{k+1}\mathrm{j}_k}, \mathbf{S}_{j_{k+1}\mathrm{j}_k}) & \text{for} \quad j_{k+1} \neq 0 \end{cases}$$

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Exercise 9.1 Show that this recursion formulae for calculating the decision function is true.

Sequential Track Extraction: Discussion

- $\mathsf{LR}(k)$ is given by a growing number of summands, each related to a particular interpretation history. The tuple $\{\lambda_{\mathbf{j}_k}, \mathbf{x}_{\mathbf{j}_k}, \mathbf{P}_{\mathbf{j}_k}\}$ is called a *sub-track*.

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- For mitigating growing memory problems all approximations discussed for track maintenance can be used if they do not significantly affect $\mathsf{LR}(k)$:

  – *individual gating:* Exclude data not likely to be associated.

  – *pruning:* Kill sub-tacks contributing marginally to the test function.

  – *local combining:* Merge similar sub tracks:

$$\{\lambda_i, \mathbf{x}_i, \mathbf{P}_i\}_i \to \{\lambda, \mathbf{x}, \mathbf{P}\} \qquad \text{with:} \quad \lambda = \textstyle\sum_i \lambda_i,$$

$$\mathbf{x} = \tfrac{1}{\lambda}\textstyle\sum_i \lambda_i \mathbf{x}_i, \quad \mathbf{P} = \tfrac{1}{\lambda}\textstyle\sum_i \lambda_i[\mathbf{P}_i + (\mathbf{x}_i - \mathbf{x})(\ldots)^\top].$$

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- The LR test ends with a decision in favor of or against the hypotheses: h_0 (no target) or h_1 (target existing). Intuitive interpretation of the thresholds!

track extraction at t_k: Decide in favor of h_1!

initiation of pdf iteration (track maintenance):

Normalize coefficients $\lambda_{\mathbf{j}_k}$: $p_{\mathbf{j}_k} = \frac{\lambda_{\mathbf{j}_k}}{\sum_{\mathbf{j}_k} \lambda_{\mathbf{j}_k}}$!

$$\boxed{(\lambda_{\mathbf{j}_k}, \mathbf{x}_{\mathbf{j}_k}, \mathbf{P}_{\mathbf{j}_k}) \quad \to \quad p(\mathbf{x}_k|\mathcal{Z}^k) = \sum_{\mathbf{j}_k} p_{\mathbf{j}_k}\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{\mathbf{j}_k},\, \mathbf{P}_{\mathbf{j}_k}\big)}$$

Continue track extraction with the remaining sensor data!

sequential LR test for track monitoring:

After deciding in favor of h_1 reset $\mathrm{LR}(0) = 1$! Calculate $\mathrm{LR}(k)$ from $p(\mathbf{x}_k|\mathcal{Z}^k)$!

track confirmation: $\mathrm{LR}(k) > \frac{P_1}{P_0}$: reset $\mathrm{LR}(0) = 1$!

track deletion: $\mathrm{LR}(k) < \frac{1-P_1}{1-P_0}$; ev. track re-initiation

DEMONSTRATION (simulated)

DEMONSTRATION (simulated)

Exercise 9.2 (voluntary)

Simulate a detection process with a given P_D, target measurements with a given $\mathbf{R}$, a detection process with a given P_D and realize the track extraction procedure.

ABRAHAM WALD (1902-1950)   Austro-Hungarian mathematician who contributed to decision theory, geometry, and econometrics; founded the theory of economic equilibria in Oskar Morgenstern's institut in Vienna: “Berechnung der Ausschaltung von Saisonschwankungen” (Springer Verlag, 1936) the basis of Game Theory: Morgenstern, John von Neumann, John Forbes Nash (1994: Nobel price with Reinhard Selten, Bonn University) $\rightarrow$ sensor management! Founder of statistical sequential analysis in WW II. 1950 plenary talk at the International Congress of Mathematicians ICM, Cambridge (Mass.): “Basic ideas of a general theory of statistical decision rules” (1900: Hilbert's 23 Problems).

Student and friend: Jacob Wolfowitz (statistician, information theory), classical text book: “Coding Theorems of Information Theory” (1978). Posthumous attack by Ronald Fisher: “an incompetent book on statistics”, passionately defended by Jerzy Neyman as imminent a statistician as Fisher.