Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction:

$$\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering:

$$\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

retrodiction:

$$\mathcal{N}(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k})$$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

$$p(\mathbf{x}_l|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, \frac{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)\, p(\mathbf{x}_l|\,\mathcal{Z}^l)}{\int d\mathbf{x}_l\, \underbrace{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)}_{\text{dynamics model}}\, \underbrace{p(\mathbf{x}_l|\,\mathcal{Z}^l)}_{\text{filtering } t_l}}\, \underbrace{p(\mathbf{x}_{l+1}|\,\mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

- $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ retrodiction: last iteration step
- $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ dynamic object behavior
- $p(\mathbf{x}_l|\mathcal{Z}^l)$ filtering at the time considered
- GAUSSians, GAUSSian mixtures: Exploit product formula!
- linear GAUSSian likelihood/dynamics: Rauch-Tung-Striebel smoothing

Fraunhofer FKIE

Sensor data: range, azimuth, range-rate

Coordinates: Sensor data → *polar*, object evolution → *Cartesian*

Dynamics system: $\mathbf{x}^d = (x, y, \dot{x}, \dot{y})$

$$p\left(\mathbf{x}^d_{k-1} \middle| \mathcal{Z}^{k-1}\right) \xrightarrow[\text{Dynamics}]{p\left(\mathbf{x}^d_k \middle| \mathbf{x}^d_{k-1}\right)} p\left(\mathbf{x}^d_k \middle| \mathcal{Z}^{k-1}\right) \qquad p\left(\mathbf{x}^d_k \middle| \mathcal{Z}^{k}\right)$$

$$\uparrow \mathbf{t}_{d\leftarrow s} \qquad \downarrow \mathbf{t}_{s\leftarrow d} \qquad \uparrow \mathbf{t}_{d\leftarrow s}$$

Sensor system: $\mathbf{x}^s = (r, \varphi, \dot{r}, \dot{\varphi})$

$$p\left(\mathbf{x}^s_{k-1} \middle| \mathcal{Z}^{k-1}\right) \qquad p\left(\mathbf{x}^s_k \middle| \mathcal{Z}^{k-1}\right) \xrightarrow[\text{Sensor}]{p\left(\mathbf{z}_k \middle| \mathbf{x}^s_k\right)} p\left(\mathbf{x}^s_k \middle| \mathcal{Z}^{k}\right)$$

scan $k-1$ scan k scan k

non-linear coordinate transformations:

$$\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s] = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{pmatrix} \qquad \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d] = \begin{pmatrix} r \\ \varphi \\ \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan y/x \\ (x\dot{x}+y\dot{y})/\sqrt{x^2+y^2} \\ (x\dot{y}-y\dot{x})/(x^2+y^2) \end{pmatrix}$$

Fraunhofer FKIE

Extended *Kalman* filter: be wise - linearize!

non-lineare transformations: Taylor expansion up to 1st order

around $\mathbf{x}^s_{k|k}$ (filtering): $\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_k] \approx \mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]\,(\mathbf{x}^s_k - \mathbf{x}^s_{k|k})$

mit: $\mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] = \partial \mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] / \partial \mathbf{x}^s_{k|k}$ (Jacobian)

around $\mathbf{x}^d_{k|k-1}$ (Prediction): $\mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_k] \approx \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_{k|k-1}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}]\,(\mathbf{x}^d_k - \mathbf{x}^d_{k|k-1})$

with: $\mathbf{T}_{s\leftarrow d} = \partial \mathbf{t}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}] / \partial \mathbf{x}^d_{k|k-1}$

affine transformation of Gaussian random variables:

$$\mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{X}\big) \xrightarrow{y=\mathbf{a}+\mathbf{A}x} \mathcal{N}\big(y;\, \mathbf{a}+\mathbf{A}\mathbf{x},\, \mathbf{A}\mathbf{X}\mathbf{A}^\top\big)$$

Fraunhofer FKIE

Recapitulation: *Expected* Measurements

innovation statistics, expectation gates, gating

$$p(\mathbf{z}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\; p(\mathbf{z}_k, \mathbf{x}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\; p(\mathbf{z}_k|\,\mathbf{x}_k)\; p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})$$

$$= \int d\mathbf{x}_k\; \underbrace{\mathcal{N}\big(\mathbf{z}_k;\; \mathbf{H}_k\mathbf{x}_k,\; \mathbf{R}_k\big)}_{\text{likelihood: sensor model}}\; \underbrace{\mathcal{N}\big(\mathbf{x}_k;\; \mathbf{x}_{k|k-1},\; \mathbf{P}_{k|k-1}\big)}_{\text{prediction at time } t_k}$$

$$= \mathcal{N}\big(\mathbf{z}_k;\; \mathbf{H}_k\mathbf{x}_{k|k-1},\; \mathbf{S}_{k|k-1}\big) \qquad \text{(product formula)}$$

innovation: $\boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1},$

innovation covariance: $\mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k$

expectation gate: $\boldsymbol{\nu}_{k|k-1}^\top \mathbf{S}_{k|k-1}^{-1} \boldsymbol{\nu}_{k|k-1} \leq \lambda(P_c)$

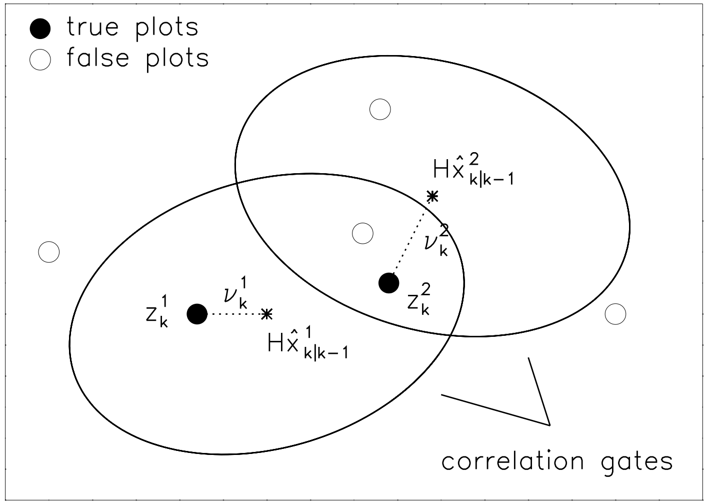
MAHALANOBIS ellipsoid containing $\mathbf{z}_k$ with certain probability P_c

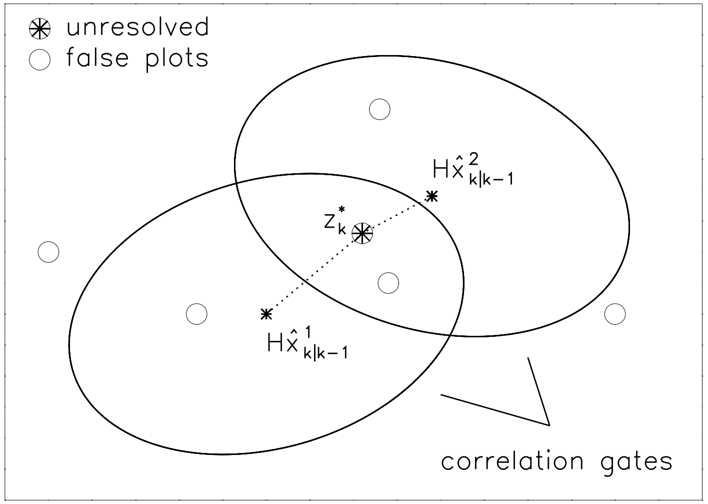
Choose $\lambda(P_c)$ (“gating parameter”) properly!

Can be looked up in a χ^2-table - discussed later!

Fraunhofer FKIE

Sensor data of uncertain origin



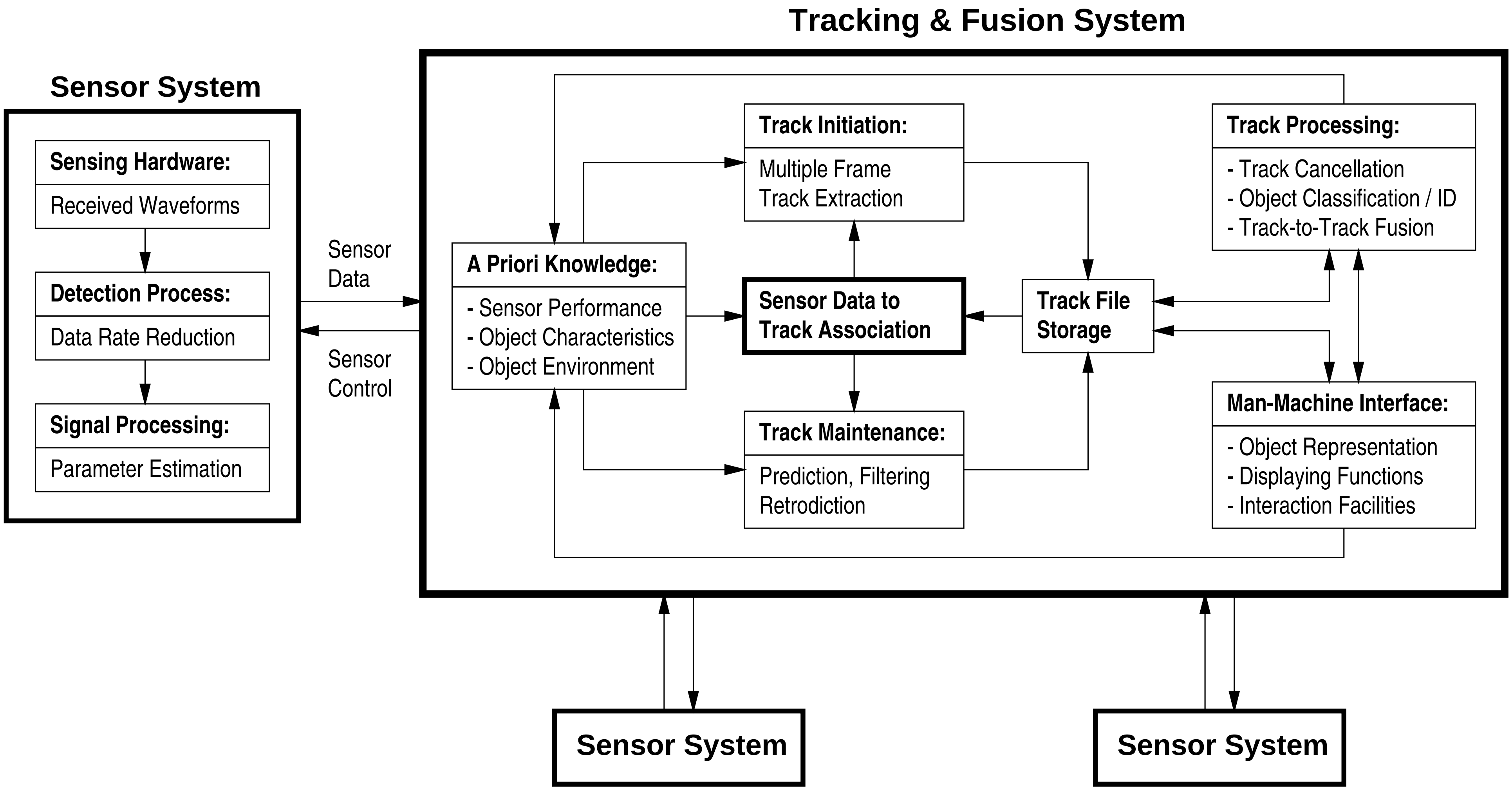


- prediction: $\mathbf{x}_{k|k-1}$, $\mathbf{P}_{k|k-1}$ (dynamics)
- innovation: $\boldsymbol{\nu}_k = \mathbf{z}_k - \mathbf{H}\mathbf{x}_{k|k-1}$, white
- Mahalanobis norm: $||\boldsymbol{\nu}_k||^2 = \boldsymbol{\nu}_k^\top \mathbf{S}_k^{-1} \boldsymbol{\nu}_k$
- expected plot: $\mathbf{z}_k \sim N(\mathbf{H}\mathbf{x}_{k|k-1}, \mathbf{S}_k)$
- $\boldsymbol{\nu}_k \sim N(0, \mathbf{S}_k)$, $\mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}$
- gating: $||\boldsymbol{\nu}_k|| < \lambda$, $P_c(\lambda)$ correlation prob.

missing/false plots, measurement errors, scan rate, agile targets: large gates

Fraunhofer FKIE

A Generic Tracking and Sensor Data Fusion System



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Description of the Detection Process

Detector: receives signals and decides on object existence

Processor: processes detected signals and produces measurements

‘D’: detector detects an object

D: object actually existent

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error of 1. kind: $P_{\mathrm{I}} = P(\neg\text{'}D\text{'}|D)$

error of 2. kind: $P_{\mathrm{II}} = P(\text{'}D\text{'}|\neg D)$

measure of detection performance: $P_D = P(\text{'}D\text{'}|D)$

detector properties characterized by two parameters:

– detection probability $P_D = 1 - P_{\mathrm{I}}$

– false alarm probability $P_F = P_{\mathrm{II}}$

Fraunhofer FKIE

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detector properties characterized by two parameters:

- detection probability $P_D = 1 - P_{\mathrm{I}}$
- false alarm probability $P_F = P_{\mathrm{II}}$

example (Swerling I model): $P_D = P_D(P_F, \text{SNR}) = P_F^{1/(1+\text{SNR})}$

detector design: Maximize detection probability P_D
for a given, predefined false alarm probability P_F!

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ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

$n_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$!

- E_0: the object was not detected; n_k false data in the Field of View (FoV)
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Consider the interpretations in the likelihood function $p(Z_k, n_k|\mathbf{x}_k)$!

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$$= p(Z_k, n_k|\neg D, \mathbf{x}_k) \underbrace{P(\neg D|\mathbf{x}_k)}_{=1-P_D} + p(Z_k, n_k|D, \mathbf{x}_k) \underbrace{P(D|\mathbf{x}_k)}_{=P_D}$$

sensor parameter: detection probability P_D

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$$= p(Z_k, n_k|\neg D, \mathbf{x}_k)\, P(\neg D|\mathbf{x}_k) + p(Z_k, n_k|D, \mathbf{x}_k)\, p(D|\mathbf{x}_k)$$

$$= \underbrace{p(Z_k|n_k, \neg D, \mathbf{x}_k)}_{=|\text{FoV}|^{-n_k}}\, \underbrace{p(n_k|\neg D, \mathbf{x}_k)}_{=p_F(n_k)}\, (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, \mathbf{x}_k)$$

false measurements: Poisson distributed in #, uniformly distributed in the FoV

Fraunhofer FKIE

Modeling of False Measurements (FM)

- **Probability of having n FM:** $p_F(n)$
  - mean number of FM in the 'Field of View' (FoV) of a sensor:

    $\bar{n} = \rho_F|\mathrm{FoV}|$, false measurement density ρ_F (perhaps not constant)

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    expectation: $\mathbb{E}[n] = \bar{n}$, variance: $\mathbb{V}[n] = \bar{n}$

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normalization: $$\sum_{n=0}^{\infty} p_F(n) = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = \mathrm{e}^{-\bar{n}}\, \mathrm{e}^{\bar{n}} = 1$$

expectation: $$\mathbb{E}[n] = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} n\frac{\bar{n}^n}{n!} = \mathrm{e}^{-\bar{n}} \sum_{n=1}^{\infty} n\frac{\bar{n}^n}{n!} = \bar{n}\mathrm{e}^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^{n-1}}{(n-1)!} = \bar{n}$$

variance: $$\mathbb{V}[n] = \mathbb{E}[(n-\bar{n})^2] = \mathbb{E}[n^2] - \bar{n}^2 = \ldots \text{exercise!} \ldots = \bar{n}$$

Modeling of False Measurements (FM)

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- **Distribution of FM in the Field of View:** $p(\mathbf{z}_1^f, \dots, \mathbf{z}_n^f | \textbf{FoV})$
  - FM mutually independent: $p(\mathbf{z}_1^f, \dots, \mathbf{z}_n^f | \mathsf{FoV}) = \prod_{i=1}^{n} p(\mathbf{z}_i^f | \mathsf{FoV})$

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Modeling of False Measurements (FM)

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  - uniformly distributed in the FoV: $p(\mathbf{z}_i^f|\text{FoV}) = |\text{FoV}|^{-1}$ (often!)

Fraunhofer FKIE

ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

$n_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$!

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$$p(Z_k, n_k|\mathbf{x}_k) = p(Z_k, n_k, \neg D|\mathbf{x}_k) + p(Z_k, n_k, D|\mathbf{x}_k) \qquad D = \text{"object was detected"}$$

$$= p(Z_k, n_k|\neg D, \mathbf{x}_k)\, P(\neg D|\mathbf{x}_k) + p(Z_k, n_k|D, \mathbf{x}_k)\, p(D|\mathbf{x}_k)$$

$$= p(Z_k|n_k, \neg D, \mathbf{x}_k)\, p(n_k|\neg D, \mathbf{x}_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, \mathbf{x}_k)$$

$$= |\text{FoV}|^{-n_k}\, p_F(n_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} \underbrace{p(Z_k|n_k, j, D, \mathbf{x}_k)}_{|\text{FoV}|^{-(n_k-1)} N(\mathbf{z}_k^j; \mathbf{H}\mathbf{x}_k, \mathbf{R})} \underbrace{p(j|n_k, D)}_{=1/n_k} \underbrace{p(n_k|D)}_{=p_F(n_k-1)}$$

Insert Poisson distribution: $p_F(n_k) = \frac{(\rho_F|\text{FoV}|)^{-n_k}}{n_k!}\, \mathrm{e}^{-\rho_F|\text{FoV}|}$

Fraunhofer FKIE

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$$= |\text{FoV}|^{-n_k}\, p_F(n_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k|n_k, j, D, \mathbf{x}_k)\, p(j|n_k, D)\, p(n_k|D)$$

$$= \frac{\mathrm{e}^{-\rho_F |\text{FoV}|}}{n_k!}\, \rho_F^{n_k - 1} \Big((1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big) \Big)$$

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Likelihood Functions

The likelihood function answers the question:

What does the sensor tell about the state $\mathbf{x}$ of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1$, $\rho_F = 0$

  at each time one measurement: $$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k; \mathbf{H}\mathbf{x}_k, \mathbf{R})$$

- **real conditions, one object:** $P_D < 1$, $\rho_F > 0$

  at each time n_k measurements $Z_k = \{\mathbf{z}_k^1, \ldots, \mathbf{z}_k^{n_k}\}$!

$$p(Z_k, n_k|\mathbf{x}_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big)$$

PDAF Filter: formally analogous to Kalman Filter

Filtering (scan $k-1$): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})$ (→ initiation)

prediction (scan k): $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$ (like Kalman)

Filtering (scan k): $p(\mathbf{x}_k|\mathcal{Z}^k) \approx \sum_{j=0}^{m_k} p_k^j \, \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}^j, \mathbf{P}_{k|k}^j)$

BLACKBOARD!

Fraunhofer FKIE

PDAF Filter: formally analogous to Kalman Filter

Filtering (scan $k-1$): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})$ ($\to$ initiation)

prediction (scan k): $p(\mathbf{x}_{k}|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_{k};\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$ (like Kalman)

Filtering (scan k): $p(\mathbf{x}_{k}|\mathcal{Z}^{k}) \approx \sum_{j=0}^{m_k} p_k^j \, \mathcal{N}(\mathbf{x}_{k};\, \mathbf{x}^j_{k|k}, \mathbf{P}^j_{k|k})$ $\approx \mathcal{N}(\mathbf{x}_{k};\, \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

$$\mathbf{x}^j_{k|k} = \begin{cases} \mathbf{x}_{k|k-1} & j=0 \\ \mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}^j_k & j\neq 0 \end{cases} \qquad \mathbf{P}^j_{k|k} = \begin{cases} \mathbf{P}_{k|k-1} & j=0 \\ \mathbf{P}_{k|k-1} - \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top & j\neq 0 \end{cases}$$

$$\boldsymbol{\nu}^j_k = \underbrace{\mathbf{z}^j_k - \mathbf{H}\mathbf{x}_k}_{\text{innovation}}, \qquad \mathbf{W}_k = \underbrace{\mathbf{P}_{k|k-1}\mathbf{H}^\top \mathbf{S}_k^{-1}}_{\text{gain matrix}}, \qquad \mathbf{S}_k = \underbrace{\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}_k}_{\text{innovation covariance}}$$

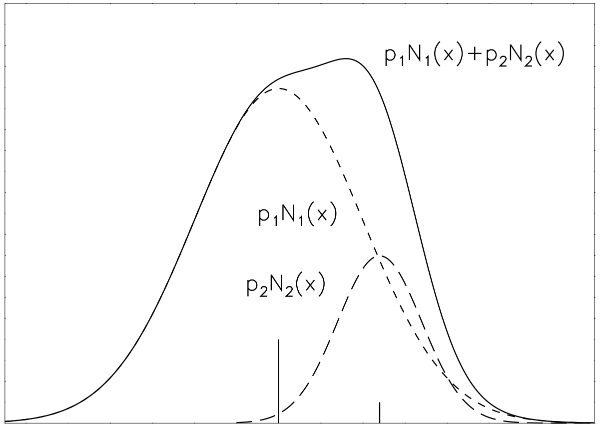
$$p^j_k = \frac{p^{j*}_k}{\underbrace{\sum_j p^{j*}_k}_{\text{Gewichte}}}, \qquad p^{j*}_k = \begin{cases} (1-P_D)\,\rho_F & j=0 \\ \frac{P_D}{\sqrt{|2\pi \mathbf{S}_k|}}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{\nu}_{H_k}^\top \mathbf{S}_k^{-1} \boldsymbol{\nu}_{H_k}} & j\neq 0 \end{cases}$$

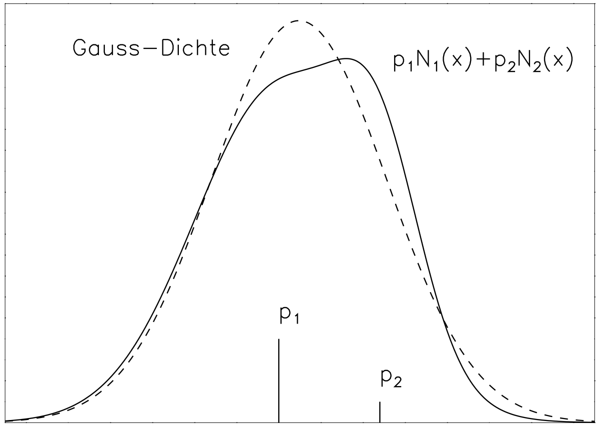
Fraunhofer FKIE

Moment Matching: Approximate an arbitrary pdf

$p(x)$ with $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{C}[x] = \mathbf{P}$ by $p(x) \approx \mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{P}\big)$!

here especially: $p(x) = \sum_H p_H \mathcal{N}(x; \mathbf{x}_H, \mathbf{P}_H)$ (normal mixtures)





$$\mathbf{x} = \sum_H p_H \mathbf{x}_H$$

$$\mathbf{P} = \sum_H p_H \left\{\mathbf{P}_H + \overbrace{(\mathbf{x}_H - \mathbf{x})(\mathbf{x}_H - \mathbf{x})^\top}^{\text{spread term}}\right\}$$

Fraunhofer FKIE

Second-order Approximation of the Mixture Density:

$$\sum_{j=1}^{m_k} p_k^j \, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k}^j,\, \mathbf{P}_{k|k}^j\big) \approx \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$$

mit:

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \, \mathbf{x}_{k|k}^j$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \, \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)$$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j, \qquad \mathbf{x}_{k|k}^0 = \mathbf{x}_{k|k-1},\ \mathbf{x}_{k|k}^j = \mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \Big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top \Big)$$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

$$= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \big(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\big)$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)$$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

$$= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \big(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\big)$$

$$= \mathbf{x}_{k|k-1} \underbrace{\left(p_k^0 + \sum_{j=1}^{m_k} p_k^j\right)}_{=1!} + \mathbf{W}_k \underbrace{\sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j}_{\text{mean!}}$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)$$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

$$= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \big(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\big) = \mathbf{x}_{k|k-1} + \mathbf{W}_k\, \boldsymbol{\nu}_k$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)$$

***Combined* Innovation:** $\boldsymbol{\nu}_k = \sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

$$= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \left(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\right) = \mathbf{x}_{k|k-1} + \mathbf{W}_k \, \boldsymbol{\nu}_k$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \left(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\right), \qquad \mathbf{P}_{k|k}^0 = \mathbf{P}_{k|k-1}, \; \mathbf{P}_{k|k}^j = \mathbf{P}_{k|k-1} - \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top$$

$$= \mathbf{P}_{k|k-1} - \sum_{j=1}^{m_k} p_k^j \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top + \sum_{j=1}^{m_k} p_k^j \mathbf{W}_k (\boldsymbol{\nu}_k^j - \boldsymbol{\nu}_k)(\boldsymbol{\nu}_k^j - \boldsymbol{\nu}_k)^\top \mathbf{W}_k^\top$$

***Combined* Innovation:** $\boldsymbol{\nu}_k = \sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j$

$$\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j$$

$$= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \big(\mathbf{x}_{k|k-1} + \mathbf{W}_k \boldsymbol{\nu}_k^j\big) = \mathbf{x}_{k|k-1} + \mathbf{W}_k \, \boldsymbol{\nu}_k$$

$$\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_k^j \big(\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top\big)$$

$$= \mathbf{P}_{k|k-1} - \sum_{j=1}^{m_k} p_k^j \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top + \sum_{j=1}^{m_k} p_k^j \mathbf{W}_k (\boldsymbol{\nu}_k^j - \boldsymbol{\nu}_k)(\boldsymbol{\nu}_k^j - \boldsymbol{\nu}_k)^\top \mathbf{W}_k^\top$$

$$= \mathbf{P}_{k|k-1} - (1 - p_k^0) \mathbf{W}_k \mathbf{S}_k \mathbf{W}_k^\top + \mathbf{W}_k \Big[\sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j \boldsymbol{\nu}_k^{j\top} - \boldsymbol{\nu}_k \boldsymbol{\nu}_k{}^\top \Big] \mathbf{W}_k^\top$$

***Combined* Innovation:** $\boldsymbol{\nu}_k = \sum_{j=1}^{m_k} p_k^j \boldsymbol{\nu}_k^j$

PDAF Filter: formally analog to Kalman Filter

Filtering (scan $k-1$): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \approx= \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})$ (→ initiation)

prediction (scan k): $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$ (like Kalman)

Filtering (scan k): $p(\mathbf{x}_k|\mathcal{Z}^{k}) \approx \sum_{j=0}^{m_k} p_k^j \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}^j, \mathbf{P}_{k|k}^j) \approx \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

$$\boldsymbol{\nu}_k = \boxed{\textstyle\sum_{j=0}^{m_k} p_k^j\, \boldsymbol{\nu}_k^j}, \quad \boldsymbol{\nu}_k^j = \mathbf{z}_k^j - \mathbf{H}\mathbf{x}_{k|k-1}$$

combined innovation

$$\mathbf{W}_k = \mathbf{P}_{k|k-1}\mathbf{H}^\top\mathbf{S}_k^{-1}, \quad \mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}_k$$

Kalman gain matrix

$$p_k^j = p_k^{i*} \,/\, \textstyle\sum_j p_k^{j*}, \quad p_k^{j*} = \begin{cases} (1-P_D)\,\rho_F \\ \frac{P_D}{\sqrt{|2\pi\mathbf{S}_{H_k}|}}\, \mathrm{e}^{-\frac{1}{2}\boldsymbol{\nu}_{H_k}^\top \mathbf{S}_{H_k}\boldsymbol{\nu}_{H_k}} \end{cases}$$

weighting factors

$$\mathbf{x}_k = \mathbf{x}_{k|k-1} + \mathbf{W}_k\,\boldsymbol{\nu}_k$$

(Filtering Update: Kalman)

$$\mathbf{P}_k = \mathbf{P}_{k|k-1} - (1-p_k^0)\,\mathbf{W}_k\mathbf{S}\mathbf{W}_k^\top$$

(Kalman part)

$$+\, \mathbf{W}_k\Big\{\textstyle\sum_{j=0}^{m_k} p_k^j\, \boldsymbol{\nu}_k^j\boldsymbol{\nu}_k^{j\top} - \boldsymbol{\nu}_k\boldsymbol{\nu}_k{}^\top\Big\}\mathbf{W}_k^\top$$

(Spread of Innovations)

PDAF: Characteristic Properties

- filtering: processing of *combined innovation*
- *all data* Z_k in the gate are considered
- p_i data dependent! Update *not linear*
- missing measurement: $\mathbf{P}_{k|k-1}$ with weight p_0
- *"usual"* Kalman covarianve according to $(1 - p_0)$
- Spread *positively semidefinite:* larger covariance
- therefore: *data driven adaptivity*
- *non linear estimator:* data dependent error
- Performance prediction *only via simulations*

Multimodality is lost! What about multiple sensor data?