Recapitulation: A popular model for object evolutions

Piecewise Constant White Acceleration Model

$$p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\Big(\mathbf{x}_k;\; \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\Big)$$

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

Consider state vectors $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$ (position, velocity, acceleration)

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\mathbf{I} & \Delta T_k\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^2\,\mathbf{I} & \Delta T_k\mathbf{I} & \mathbf{I} \end{pmatrix}$$

with $\Delta T_k = t_k - t_{k-1}$. Reasonable choice: $\frac{1}{2}v_{\text{max}}/v_{\text{max}} \leq \Sigma_k \leq v_{\text{max}}/q_{\text{max}}$

Another, rather realistic model (van Keuk):

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} & \frac{1}{2}(t_k - t_{k-1})^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-(t_k - t_{k-1})/\theta}\,\mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \mathrm{diag}[1,1,1]$$

$$\mathbf{D}_{k|k-1} = \Sigma^2 (1 - \mathrm{e}^{-2(t_k - t_{k-1})/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{O} = \mathrm{diag}[0,0,0]$$

maneuver correlation time θ (z.B. 60 s), *limiting acceleration* Σ (z.B. 2 g)

There are many different evolution models adapted to particular problems!

Exercise 5.1 (voluntary!)

Show for the acceleration process:

$$\mathbb{E}[\ddot{\mathbf{r}}_k] = 0, \quad \mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top] = \Sigma^2\, \mathrm{e}^{-(t_k - t_l)/\theta}\, \mathbf{I}, \; l \leq k$$

$\mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top]$ is called ‘auto correlation function’.

Idealized measurement process

- **linear measurement equation:**

  $$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{u}_k, \quad p(\mathbf{u}_k) = \mathcal{N}\big(\mathbf{u}_k;\, \mathbf{o},\, \mathbf{R}_k\big)$$

  – to be measured: *linear* functions of the object state
  – measurement error: biasfree, Gaussian distrib.
    independent for different t_k
  – $\mathbf{y}_k = \mathbf{z}_k - \mathbf{H}_k \mathbf{x}_k$ has the pdf: $p(\mathbf{y}_k) = p(\mathbf{u}_k)$

- **Approach for the requested pdf ('likelihood fkt.):**

  $$\boxed{p\big(\mathbf{z}_k | \mathbf{x}_k\big) = \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k\big)}$$

- **Example: position measurement**

  $$\mathbf{H}_k = (\mathbf{I}, \mathbf{O}, \mathbf{O}), \quad \mathbf{H}_k \mathbf{x}_k = \mathbf{r}_k$$

  $\mathbf{R}_k$: measurement error covariance matrix

  possibly depending on the sensor-to-target geometry

a first remark on initiation: $p(\mathbf{x}_0|\mathcal{Z}^0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, $\mathbf{P}_{0|0}$ 'large'

$$\mathbf{x}_{0|0} = \begin{pmatrix} \mathbf{r}_{0|0} \\ \dot{\mathbf{r}}_{0|0} \\ \ddot{\mathbf{r}}_{0|0} \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{P}_{0|0} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (v_{max})^2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (q_{max})^2\mathbf{1} \end{pmatrix}$$

position information: first measurement $\mathbf{z}_0$, ignorance = measurement error $\mathbf{R}$!

ignorance on velocity: sphere with radius v_{max} around zero
(= no information on direction, but on 'limits')

ignorance on acceleration: sphere with radius q_{max} around zero

Fraunhofer FKIE

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$ 'KALMAN gain matrix'

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$p(\mathbf{z}_k^1, \mathbf{z}_k^2, \ldots \mathbf{z}_k^{S_k} | \mathbf{x}_k) = \prod_{s=1}^{S_k} p(\mathbf{z}_k^s | \mathbf{x}_k) \qquad \text{independent sensors}$$

$$= \prod_{s=1}^{S_k} \mathcal{N}\left(\mathbf{z}_k^s;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^s\right) \propto \mathcal{N}\left(\mathbf{z}_k;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k\right)$$

with: $$\mathbf{z}_k = \mathbf{R}_k \sum_{s=1}^{S_k} (\mathbf{R}_k^s)^{-1} \mathbf{z}_k^s, \quad \mathbf{R}_k = \left(\sum_{s=1}^{S_k} (\mathbf{R}_k^s)^{-1} \right)^{-1}$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big)\, \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big)\; \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

$$p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big) = \frac{p(Z_k, \dots, Z_{l+1}|\mathbf{x}_{l+1}, \mathbf{x}_l, \mathcal{Z}^l)\; p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l\big)}{\int d\mathbf{x}_l\; p(Z_k, \dots, Z_{l+1}|\mathbf{x}_{l+1}, \mathbf{x}_l, \mathcal{Z}^l)\; p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l\big)} = p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l\big)$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big)\; \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

$$p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k\big) = p\big(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l\big) = \frac{p\big(\mathbf{x}_{l+1}|\,\mathbf{x}_l\big)\; p\big(\mathbf{x}_l|\,\mathcal{Z}^l\big)}{\int d\mathbf{x}_l\;\; \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathbf{x}_l\big)}_{\text{dynamics model}}\;\; \underbrace{p\big(\mathbf{x}_l|\,\mathcal{Z}^l\big)}_{\text{filtering } t_l}}$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p(\mathbf{x}_l|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, \frac{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)\; p(\mathbf{x}_l|\,\mathcal{Z}^l)}{\int d\mathbf{x}_l\; \underbrace{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)}_{\text{dynamics model}}\; \underbrace{p(\mathbf{x}_l|\,\mathcal{Z}^l)}_{\text{filtering } t_l}}\; \underbrace{p(\mathbf{x}_{l+1}|\,\mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

- $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ retrodiction: last iteration step
- $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ dynamic object behavior
- $p(\mathbf{x}_l|\mathcal{Z}^l)$ filtering at the time considered
- GAUSSians, GAUSSian mixtures: Exploit product formula!
- linear GAUSSian likelihood/dynamics: Rauch-Tung-Striebel smoothing

Exercise 6.1 Derive the *Rauch-Tung-Striebel* formulae

by using the Kalman filter assumptions

and the product formula (twice)!

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$

$$\begin{aligned} \mathbf{x}_{l|k} &= \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^{\top}\mathbf{P}_{l+1|l}^{-1} \\ \mathbf{P}_{l|k} &= \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^{\top} \end{aligned}$$

Kalman filter: linear Gaussian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'Kalman gain matrix'}$$

retrodiction: $\mathcal{N}(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k})$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$

Exercise 6.2 Implement the *Rauch-Tung-Striebel* formulae in your simulator!

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big) \end{aligned}$$

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big) \end{aligned}$$

Looks like a Kalman filtering update!

Fraunhofer FKIE

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \boldsymbol{\Delta}_{l+\theta|l+1}\big)$$

$$\begin{aligned} \mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\ &= \mathbf{x}_{l+\theta|l} - \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \end{aligned}$$

$$\boldsymbol{\Delta}_{l+\theta|l+1} = \mathbf{P}_{l+\theta|l} - \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\boldsymbol{\Phi}^\top_{l+\theta|l+1}$$

$$\begin{aligned} \boldsymbol{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\ \mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}. \end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \boldsymbol{\Delta}_{l+\theta|l+1}\big)$$

$$= \mathcal{N}\big(\mathbf{b}_{l+\theta|l+1};\, \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \boldsymbol{\Delta}_{l+\theta|l+1}\big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} \\
\boldsymbol{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \boldsymbol{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\boldsymbol{\Phi}^\top_{l+\theta|l+1}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big)$$
$$= \mathcal{N}\Big(\mathbf{b}_{l+\theta|l+1};\, \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} \\
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}^{\top}_{l+\theta|l+1}
\end{aligned}$$

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Looks like prediction!

$$= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{x}_{l+\theta|k},\, \mathbf{x}_{l+\theta|k}\Big)$$

$$\begin{aligned}
\mathbf{x}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\left(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}\right) \\
\mathbf{P}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\left(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l}\right)\mathbf{\Phi}^{\top}_{l+\theta|l+1}
\end{aligned}$$

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\ \mathbf{x}_{0|0},\ \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}_{k-1|k-1},\ \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\ \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\ \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k},\ \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\ \mathbf{x}_{l|k},\ \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\ \mathbf{x}_{l+1|k},\ \mathbf{P}_{l+1|k}\big)$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$

Sensor data: range, azimuth, range-rate

Coordinates: Sensor data $\to$ *polar*, object evolution $\to$ *Cartesian*

Dynamics system: $\mathbf{x}^d = (x, y, \dot{x}, \dot{y})$

Sensor system: $\mathbf{x}^s = (r, \varphi, \dot{r}, \dot{\varphi})$

	scan $k-1$		scan k		scan k
Dynamics system	$p(\mathbf{x}^d_{k-1} \mid \mathcal{Z}^{k-1})$	$\xrightarrow[\text{Dynamics}]{p(\mathbf{x}^d_k \mid \mathbf{x}^d_{k-1})}$	$p(\mathbf{x}^d_k \mid \mathcal{Z}^{k-1})$		$p(\mathbf{x}^d_k \mid \mathcal{Z}^k)$
	$\uparrow \mathbf{t}_{d\leftarrow s}$		$\downarrow \mathbf{t}_{s\leftarrow d}$		$\uparrow \mathbf{t}_{d\leftarrow s}$
Sensor system	$p(\mathbf{x}^s_{k-1} \mid \mathcal{Z}^{k-1})$		$p(\mathbf{x}^s_k \mid \mathcal{Z}^{k-1})$	$\xrightarrow[\text{Sensor}]{p(\mathbf{z}_k \mid \mathbf{x}^s_k)}$	$p(\mathbf{x}^s_k \mid \mathcal{Z}^k)$

non-linear coordinate transformations:

$$\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s] = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{pmatrix} \qquad \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d] = \begin{pmatrix} r \\ \varphi \\ \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan y/x \\ (x\dot{x}+y\dot{y})/\sqrt{x^2+y^2} \\ (x\dot{y}-y\dot{x})/(x^2+y^2) \end{pmatrix}$$

Extended *Kalman* filter: be wise - linearize!

non-lineare transformations: Taylor expansion up to 1st order

around $\mathbf{x}^s_{k|k}$ (filtering): $\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_k] \approx \mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]\,(\mathbf{x}^s_k - \mathbf{x}^s_{k|k})$

mit: $\mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]{=}\partial\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]/\partial\mathbf{x}^s_{k|k}$ (Jacobian)

around $\mathbf{x}^d_{k|k-1}$ (Prediction): $\mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_k] \approx \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_{k|k-1}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}]\,(\mathbf{x}^d_k - \mathbf{x}^d_{k|k-1})$

with: $\mathbf{T}_{s\leftarrow d}{=}\partial\mathbf{t}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}]/\partial\mathbf{x}^d_{k|k-1}$

affine transformation of Gaussian random variables:

$$\mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{X}\big) \xrightarrow{y=\mathbf{a}+\mathbf{A}x} \mathcal{N}\big(y;\, \mathbf{a}+\mathbf{A}\mathbf{x},\, \mathbf{A}\mathbf{X}\mathbf{A}^\top\big)$$

A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

$$p(\mathbf{z}_k | \mathcal{Z}^{k-1}) = \int d\mathbf{x}_k \, p(\mathbf{z}_k, \mathbf{x}_k | \mathcal{Z}^{k-1}) = \int d\mathbf{x}_k \, p(\mathbf{z}_k | \mathbf{x}_k) \, p(\mathbf{x}_k | \mathcal{Z}^{k-1})$$

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$$= \int d\mathbf{x}_k\; \underbrace{\mathcal{N}\big(\mathbf{z}_k;\; \mathbf{H}_k\mathbf{x}_k,\; \mathbf{R}_k\big)}_{\text{likelihood: sensor model}}\; \underbrace{\mathcal{N}\big(\mathbf{x}_k;\; \mathbf{x}_{k|k-1},\; \mathbf{P}_{k|k-1}\big)}_{\text{prediction at time } t_k}$$

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$$= \mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_{k|k-1},\, \mathbf{S}_{k|k-1}) \qquad \text{(product formula)}$$

innovation: $\boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1},$

innovation covariance: $\mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k$

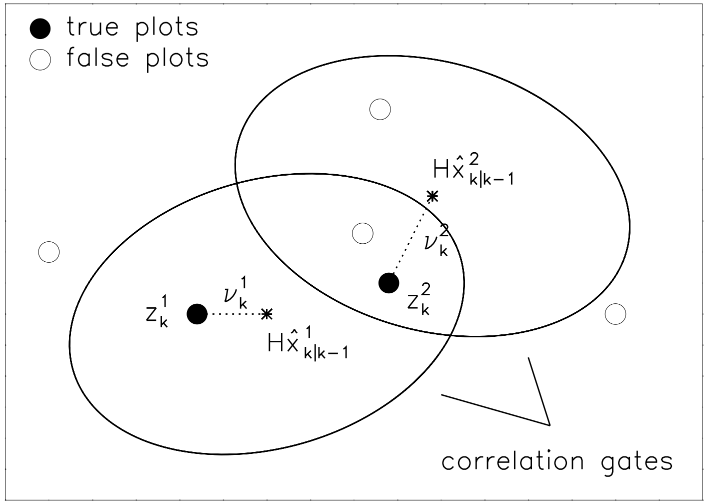
expectation gate: $\boldsymbol{\nu}_{k|k-1}^\top \mathbf{S}_{k|k-1}^{-1} \boldsymbol{\nu}_{k|k-1} \leq \lambda(P_c)$

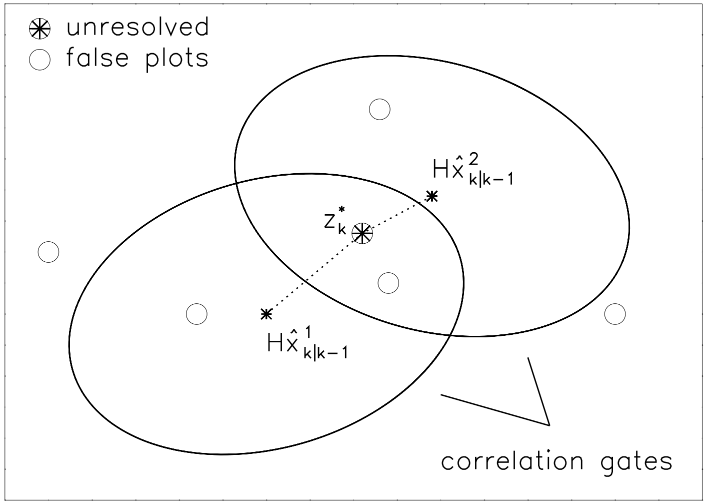
MAHALANOBIS ellipsoid containing $\mathbf{z}_k$ with certain probability P_c

Choose $\lambda(P_c)$ (“gating parameter”) properly!

Can be looked up in a χ^2-table - discussed later!

Sensor data of uncertain origin



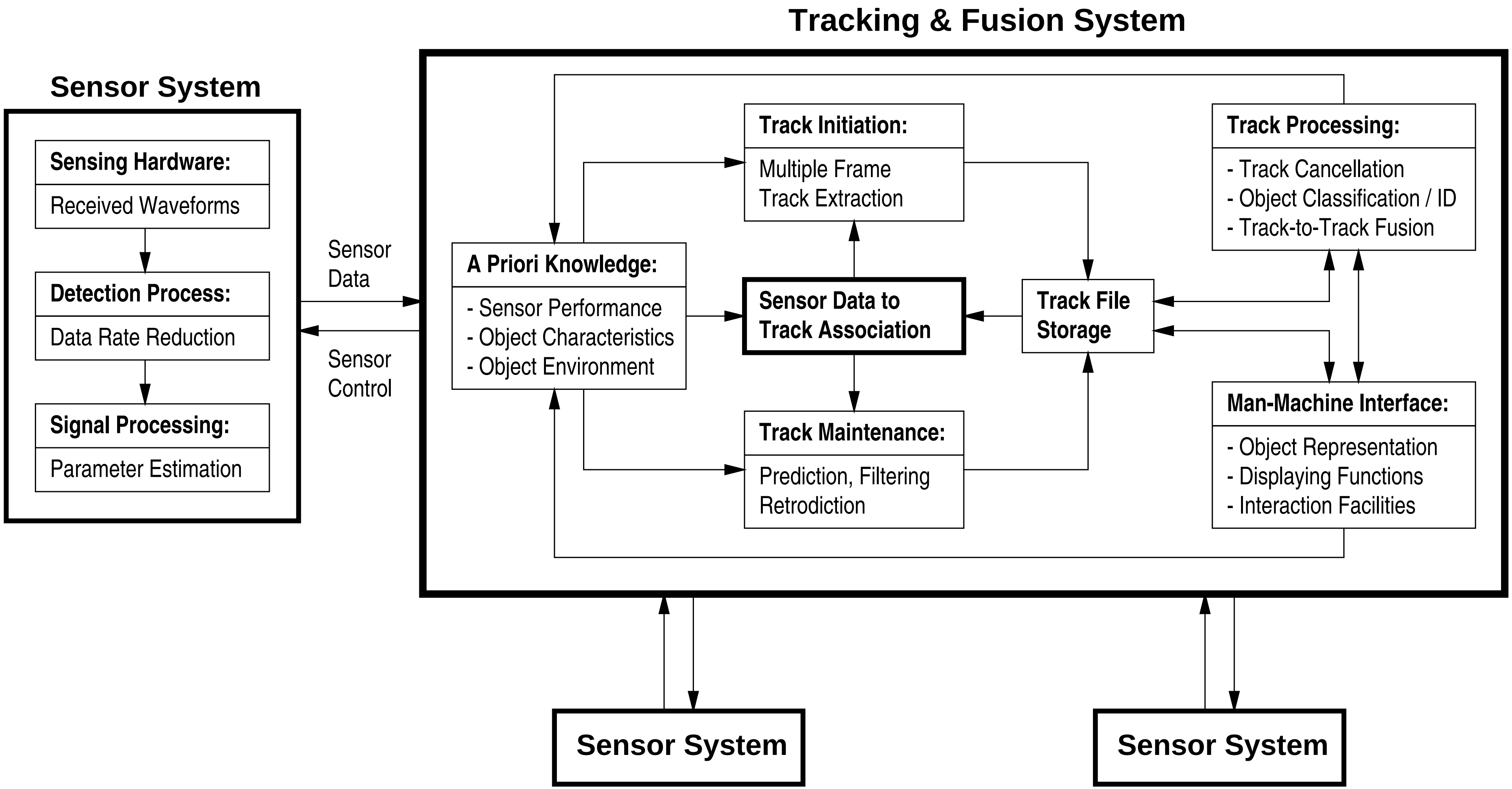


- prediction: $\mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1}$ (dynamics)
- innovation: $\boxed{\boldsymbol{\nu}_k = \mathbf{z}_k - \mathbf{H}\mathbf{x}_{k|k-1}}$, white
- Mahalanobis norm: $\boxed{||\boldsymbol{\nu}_k||^2 = \boldsymbol{\nu}_k^\top \mathbf{S}_k^{-1} \boldsymbol{\nu}_k}$
- expected plot: $\mathbf{z}_k \sim N(\mathbf{H}\mathbf{x}_{k|k-1}, \mathbf{S}_k)$
- $\boldsymbol{\nu}_k \sim N(0, \mathbf{S}_k)$, $\boxed{\mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}}$
- gating: $||\boldsymbol{\nu}_k|| < \lambda$, $P_c(\lambda)$ correlation prob.

missing/false plots, measurement errors, scan rate, agile targets: large gates

Fraunhofer FKIE

A Generic Tracking and Sensor Data Fusion System



Description of the Detection Process

Detector: receives signals and decides on object existence

Processor: processes detected signals and produces measurements

'D': detector detects an object

D: object actually existent

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error of 1. kind: $P_\mathrm{I} = P(\neg\text{'}D\text{'}|D)$

error of 2. kind: $P_\mathrm{II} = P(\text{'}D\text{'}|\neg D)$

measure of detection performance: $P_D = P(\text{'}D\text{'}|D)$

detector properties characterized by two parameters:

– detection probability $P_D = 1 - P_\mathrm{I}$
– false alarm probability $P_F = P_\mathrm{II}$

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example (Swerling I model): $P_D = P_D(P_F, \text{SNR}) = P_F^{1/(1+\text{SNR})}$

detector design: Maximize detection probability P_D
for a given, predefined false alarm probability P_F!

ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

$n_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$!

- E_0: the object was not detected; n_k false data in the Field of View (FoV)
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Consider the interpretations in the likelihood function $p(Z_k, n_k | \mathbf{x}_k)$!

Fraunhofer FKIE

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$$= p(Z_k, n_k|\neg D, \mathbf{x}_k)\ \underbrace{P(\neg D|\mathbf{x}_k)}_{=1-P_D} + p(Z_k, n_k|D, \mathbf{x}_k)\ \underbrace{P(D|\mathbf{x}_k)}_{=P_D}$$

sensor parameter: detection probability P_D

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$$= p(Z_k, n_k|\neg D, \mathbf{x}_k)\, P(\neg D|\mathbf{x}_k) + p(Z_k, n_k|D, \mathbf{x}_k)\, p(D|\mathbf{x}_k)$$

$$= \underbrace{p(Z_k|n_k, \neg D, \mathbf{x}_k)}_{=|\text{FoV}|^{-n_k}}\ \underbrace{p(n_k|\neg D, \mathbf{x}_k)}_{=p_F(n_k)}\,(1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, \mathbf{x}_k)$$

false measurements: Poisson distributed in #, uniformly distributed in the FoV

Modeling of False Measurements (FM)

- **Probability of having n FM:** $p_F(n)$
  - mean number of FM in the 'Field of View' (FoV) of a sensor:

    $\bar{n} = \rho_F|\mathsf{FoV}|$, false measurement density ρ_F (perhaps not constant)

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    expectation: $\mathbb{E}[n] = \bar{n}$, variance: $\mathbb{V}[n] = \bar{n}$

normalization: $\sum_{n=0}^{\infty} p_F(n) = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = \mathrm{e}^{-\bar{n}}\,\mathrm{e}^{\bar{n}}$ = 1

expectation: $\mathbb{E}[n] = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} n \frac{\bar{n}^n}{n!} = \mathrm{e}^{-\bar{n}} \sum_{n=1}^{\infty} n \frac{\bar{n}^n}{n!} = \bar{n}\mathrm{e}^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^{n-1}}{(n-1)!} = \bar{n}$

variance: $\mathbb{V}[n] = \mathbb{E}[(n-\bar{n})^2] = \mathbb{E}[n^2] - \bar{n}^2 = \ldots \text{exercise!} \ldots = \bar{n}$

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- **Distribution of FM in the Field of View:** $p(\mathbf{z}_1^f, \ldots, \mathbf{z}_n^f|\textbf{FoV})$

  – FM mutually independent: $p(\mathbf{z}_1^f, \ldots, \mathbf{z}_n^f|\mathsf{FoV}) = \prod_{i=1}^{n} p(\mathbf{z}_i^f|\mathsf{FoV})$

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  - uniformly distributed in the FoV: $p(\mathbf{z}_i^f|\mathsf{FoV}) = |\mathsf{FoV}|^{-1}$ (often!)

ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

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$$= p(Z_k, n_k|\neg D, \mathbf{x}_k)\, P(\neg D|\mathbf{x}_k) + p(Z_k, n_k|D, \mathbf{x}_k)\, p(D|\mathbf{x}_k)$$

$$= p(Z_k|n_k, \neg D, \mathbf{x}_k)\, p(n_k|\neg D, \mathbf{x}_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, \mathbf{x}_k)$$

$$= |\text{FoV}|^{-n_k}\, p_F(n_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} \underbrace{p(Z_k|n_k, j, D, \mathbf{x}_k)}_{|\text{FoV}|^{-(n_k-1)} N(\mathbf{z}_k^j; \mathbf{H}\mathbf{x}_k, \mathbf{R})}\ \underbrace{p(j|n_k, D)}_{=1/n_k}\ \underbrace{p(n_k|D)}_{=p_F(n_k-1)}$$

Insert Poisson distribution: $p_F(n_k) = \frac{(\rho_F|\text{FoV}|)^{-n_k}}{n_k!}\, \mathrm{e}^{-\rho_F|\text{FoV}|}$

Fraunhofer FKIE

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Consider the interpretations in the likelihood function $p(Z_k, n_k|\mathbf{x}_k)$!

$$p(Z_k, n_k|\mathbf{x}_k) = p(Z_k, n_k, \neg D|\mathbf{x}_k) + p(Z_k, n_k, D|\mathbf{x}_k) \qquad D = \text{"object was detected"}$$

$$= p(Z_k, n_k|\neg D, \mathbf{x}_k)\, P(\neg D|\mathbf{x}_k) + p(Z_k, n_k|D, \mathbf{x}_k)\, p(D|\mathbf{x}_k)$$

$$= p(Z_k|n_k, \neg D, \mathbf{x}_k)\, p(n_k|\neg D, \mathbf{x}_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, \mathbf{x}_k)$$

$$= |\text{FoV}|^{-n_k}\, p_F(n_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k|n_k, j, D, \mathbf{x}_k)\, p(j|n_k, D)\, p(n_k|D)$$

$$= \frac{e^{-\rho_F |\text{FoV}|}}{n_k!}\, \rho_F^{n_k - 1} \Big((1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big) \Big)$$

Fraunhofer FKIE

Likelihood Functions

The likelihood function answers the question:

What does the sensor tell about the state $\mathbf{x}$ of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1$**,** $\rho_F = 0$

  at each time one measurement: $p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k; \mathbf{H}\mathbf{x}_k, \mathbf{R})$

- **real conditions, one object:** $P_D < 1$**,** $\rho_F > 0$

  at each time n_k measurements $Z_k = \{\mathbf{z}_k^1, \ldots, \mathbf{z}_k^{n_k}\}$!

$$p(Z_k, n_k|\mathbf{x}_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big)$$