Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\begin{aligned} \mathbf{x}_{k|k} &= \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, & \boldsymbol{\nu}_{k|k-1} &= \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1} \\ \mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, & \mathbf{S}_{k|k-1} &= \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k \end{aligned}$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'KALMAN gain matrix'}$$

Recapitulation: A popular model for object evolutions

Piecewise Constant White Acceleration Model

$$p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)$$

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

with $\Delta T_k = t_k - t_{k-1}$. Reasonable choice: $\frac{1}{2}v_{\text{max}} \leq \Sigma_k \leq q_{\text{max}}$

Recapitulation: A popular model for object evolutions

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Consider state vectors $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$ (position, velocity, acceleration)

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\mathbf{I} & \Delta T_k\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^2\,\mathbf{I} & \Delta T_k\mathbf{I} & \mathbf{I} \end{pmatrix}$$

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Fraunhofer FKIE

Object evolution: Gauss-Markov process

- **linear evolution equation:** $\mathbf{x}_k = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{o}, \mathbf{D}_{k|k-1})$

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$\mathbf{x}_k$ is an affine transformation of a Gaussian RV $\mathbf{v}_k$ with the pdf

$p(\mathbf{v}_k) = \mathcal{N}\big(\mathbf{v}_k;\, \mathbf{o},\, \mathbf{D}_{k|k-1}\big)$. Thus also $\mathbf{x}_k$ is a Gaussian RV with:

$$\mathbb{E}[\mathbf{x}_k] = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{I}\mathbf{o} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1}$$

$$\mathbb{C}[\mathbf{x}_k] = \mathbf{I}\mathbf{D}_{k|k-1}\mathbf{I}^\top = \mathbf{D}_{k|k-1}$$

Recapitulation: Affine Transforms of GAUSSian RVs

$$\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{D}\big)$ with $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D}\to 0$$

$$= \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top + \mathbf{D}\big) \quad \text{for } \mathbf{D}\to 0; \quad \text{product formula!}$$

Also true if $\dim(\mathbf{x}) \neq \dim(\mathbf{y})$**!**

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- **Very simple example:**

  Object on a strait line: 2D state $\mathbf{x}_k = (x_k, \dot{x}_k)^\top$

  simple approach:
  $$\begin{aligned} x_k &= x_{k-1} + \Delta t\, \dot{x}_{k-1} & \Delta t = t_k - t_{k-1} \\ \dot{x}_k &= \dot{x}_{k-1} + v & v \sim N(0, D) \end{aligned}$$

  we thus have: $\mathbf{x}_k = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \mathbf{x}_{k-1} + \mathbf{v}, \quad \mathbf{v} \sim N(0, \mathbf{D}),\ \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$

Fraunhofer FKIE

Object evolution: Gauss-Markov process

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- **Requested: Markov property!**

$$p\big(\mathbf{x}_k \big| \, \mathbf{x}_{k-1}, \mathbf{x}_{k-2}, \ldots, \mathbf{x}_1\big) \overset{!}{=} p\big(\mathbf{x}_k \big| \, \mathbf{x}_{k-1}\big) \overset{!}{=} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}_{k|k-1}\big)$$

Object evolution: Gauss-Markov process

- **linear evolution equation:** $\mathbf{x}_k = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{o}, \mathbf{D}_{k|k-1})$

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therefore:
$$\begin{aligned} p(\mathbf{x}_k) &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k, \mathbf{x}_{k-1}, \dots \mathbf{x}_1) \\ &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}, \dots \mathbf{x}_1) \\ &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k|\mathbf{x}_{k-1}) \dots p(\mathbf{x}_2|\mathbf{x}_1)\, p(\mathbf{x}_1) \end{aligned}$$

- **Another, rather realistic model (van Keuk):**

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} & \frac{1}{2}(t_k - t_{k-1})^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-(t_k - t_{k-1})/\theta}\,\mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \mathrm{diag}[1, 1, 1]$$

$$\mathbf{D}_{k|k-1} = \Sigma^2 (1 - \mathrm{e}^{-2(t_k - t_{k-1})/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{O} = \mathrm{diag}[0, 0, 0]$$

There are many different evolution models adapted to particular problems!

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There are many different evolution models adapted to particular problems!

Show for the acceleration process:

$$\mathbb{E}[\ddot{\mathbf{r}}_k] = 0, \quad \mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top] = \Sigma^2\,\mathrm{e}^{-(t_k - t_l)/\theta}\,\mathbf{I},\ l \leq k$$

Exercise 5.1 (voluntary!)

θ: maneuver correlation time, Σ: limiting acceleration

$\mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top]$ is called ‘auto correlation function’.

Modelling of the Measurement Process

- **What tells the measurement about the object?**

  functional relation between measurement $\mathbf{z}_k = (z_k^1, \ldots, z_k^m)$
  at time t_k and the object state $\mathbf{x}_k = (x_k^1, \ldots, x_k^n)^\top$ at the same time

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  Example for the 2D object state $\mathbf{x}_k = (x_k^1, x_k^2)^\top$:

  $-$ x_1 component measured: $\mathbf{z}_k = \mathbf{H}\mathbf{x}_k, \quad \mathbf{H} = (\,1\;0\,)$ linear function

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  – range/distance: $\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k)$, $\mathbf{h}(\mathbf{x}_k) = \begin{pmatrix} \sqrt{(x_k^1)^2 + (x_k^2)^2} \\ \tan^{-1} x_k^2 / x_k^1 \end{pmatrix}$ non-linear!

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functional relation between measurement $\mathbf{z}_k = (z_k^1, \dots, z_k^m)$
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– range/distance: $\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k)$, $\mathbf{h}(\mathbf{x}_k) = \begin{pmatrix} \sqrt{(x_k^1)^2 + (x_k^2)^2} \\ \tan^{-1} x_k^2/x_k^1 \end{pmatrix}$ non-linear!

- **What is the quality of the measurement?**

reasonable: measurement error $\mathbf{u}_k = \mathbf{z}_k - \mathbf{H}\mathbf{x}_k$ normally distributed with:

$$\mathbb{E}[\mathbf{u}_k] = 0,\ \mathbb{C}[\mathbf{u}_k] = \begin{pmatrix} \mathbb{E}[(u_k^1)^2] & \mathbb{E}[u_k^1 u_k^2] \\ \mathbb{E}[u_k^2 u_k^1] & \mathbb{E}[(u_k^2)^2] \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \mathbb{E}[u_k^1 u_k^2] \\ \mathbb{E}[u_k^2 u_k^1] & \sigma_2^2 \end{pmatrix} = \mathbf{R}$$

uncorrelated: $\mathbb{E}[u_k^1 u_k^2] = 0$; Gaussian: uncorrelated $\Rightarrow$ independent (not i.g.!)
Standard deviations σ appear on the diagonal of the covariance matrix.

Idealisized measurement process

- **linear measurement equation:**

  $$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{u}_k, \quad p(\mathbf{u}_k) = \mathcal{N}\big(\mathbf{u}_k;\, \mathbf{o},\, \mathbf{R}_k\big)$$

  – to be measured: *linear* functions of the object state
  – measurement error: biasfree, Gaussian distrib.
  independent for different t_k
  – $\mathbf{y}_k = \mathbf{z}_k - \mathbf{H}_k \mathbf{x}_k$ has the pdf: $p(\mathbf{y}_k) = p(\mathbf{u}_k)$

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  – $\mathbf{y}_k = \mathbf{z}_k - \mathbf{H}_k \mathbf{x}_k$ has the pdf: $p(\mathbf{y}_k) = p(\mathbf{u}_k)$

- **Approach for the requested pdf ('likelihood fkt.):**

  $$\boxed{p\Big(\mathbf{z}_k \big| \, \mathbf{x}_k\Big) = \mathcal{N}\Big(\mathbf{z}_k;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k\Big)}$$

- **Example: position measurement**

  $$\mathbf{H}_k = (\mathbf{I}, \mathbf{O}, \mathbf{O}), \quad \mathbf{H}_k \mathbf{x}_k = \mathbf{r}_k$$

  $$\mathbf{R}_k = \mathrm{diag}[\sigma_x^2, \sigma_y^2, \sigma_z^2], \ \sigma_x : \text{ measurement error}$$

Fraunhofer FKIE

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\ \mathbf{x}_{0|0},\ \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\ \mathbf{x}_{k-1|k-1},\ \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\ \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\ \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\ \mathbf{x}_{k|k},\ \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$
'KALMAN gain matrix'

Exercise 5.2

In your sensor simulator, chose a sensor at position $\mathbf{r}_s$, for example $\mathbf{r}_s = (0, 0)^\top$, that produces measurements $\mathbf{z}_k$ of the Cartesian target positions $\mathbf{H}\mathbf{x}_k$ from your ground truth generator. Use the measurement covariance matrix $\mathbf{R} = \sigma_c^2 \mathrm{diag}[1, 1]$, σ_c = 50 m. Program your first Kalman filter using a constant acceleration or the van Keuk model. Visualize your results nicely! Compare the ground truth, the measurement, and the estimates!

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

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Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) = p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \qquad \text{independent sensors}$$

$$= \mathcal{N}\left(\mathbf{z}_k^1;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^1\right) \mathcal{N}\left(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\right)$$

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$$= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big)$$

A Useful Product Formula for Gaussians

$$\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\,\mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Fraunhofer FKIE

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$\begin{aligned} p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) &= p(\mathbf{z}_k^1|\mathbf{x}_k)\, p(\mathbf{z}_k^2|\mathbf{x}_k) \qquad \text{independent sensors} \\ &= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \\ &= \mathcal{N}\big(\mathbf{H}_k\mathbf{x}_k;\, \mathbf{z}_k^1,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \\ &\propto \mathcal{N}\big(\mathbf{H}_k\mathbf{x}_k;\, \underbrace{\mathbf{R}_k(\mathbf{R}_k^{1\;-1}\mathbf{z}_k^1 + \mathbf{R}_k^{2\;-1}\mathbf{z}_k^2)}_{=\mathbf{z}_k},\, \underbrace{(\mathbf{R}_k^{1\;-1} + \mathbf{R}_k^{2\;-1})^{-1}}_{=\mathbf{R}_k}\big) \\ &= \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big) \end{aligned}$$

Exercise 5.3 Generalize to the case $S_k > 2$ (induction argument)!

One possible fusion strategy:

Create a singe effective measurement
by preprocessing of individual sensor measurement!

$$\mathbf{z}_k = \mathbf{R}_k \sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1} \mathbf{z}_k^s$$ weighted arithmetic mean of measurements

$$\mathbf{R}_k = \left(\sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1} \right)^{-1}$$ harmonic mean of measuremt covariances

A typical structure for fusion equations!

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$

$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$ 'KALMAN gain matrix'

Exercise 5.4

In your sensor simulator, chose an **arbitrary number S of sensors** at positions $\mathbf{r}_s$, $s = 1, \dots, S$, produce measurements $\mathbf{z}_k^s$, $s = 1, \dots, S$, of the Cartesian target positions $\mathbf{H}\mathbf{x}_k$ from your ground truth generator. Use preprocessing both algorithms! Discuss pros & cons!

Towards real world sensors: range, azimuth data

- **Gaussian measurements in polar coordinates:**

$$\mathbf{z}_k^p = (r_k, \varphi_k)^\top, \quad \text{error covariance matrix: } \mathbf{R}^p = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **transformation into Cartesian target positions:**

$$\mathbf{z}_k^c = \mathbf{t}[\mathbf{z}_k^p] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \quad \text{A non-affin transformation!}$$

A Taylor-series approximation of $\mathbf{t}[\mathbf{z}_k^p]$ up to the first order were affin!

Towards real world sensors: range, azimuth data

- **Gaussian measurements in polar coordinates:**

$$\mathbf{z}_k^p = (r_k, \varphi_k)^\top, \quad \text{error covariance matrix: } \mathbf{R}^p = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **Taylor-approximation around:** $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:

$$\mathbf{z}_k^c = \mathbf{t}[\mathbf{z}_k^p] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

Towards real world sensors: range, azimuth data

- **Gaussian measurements in polar coordinates:**

$$\mathbf{z}_k^p = (r_k, \varphi_k)^\top, \quad \text{error covariance matrix: } \mathbf{R}^p = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **Taylor-approximation around:** $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:

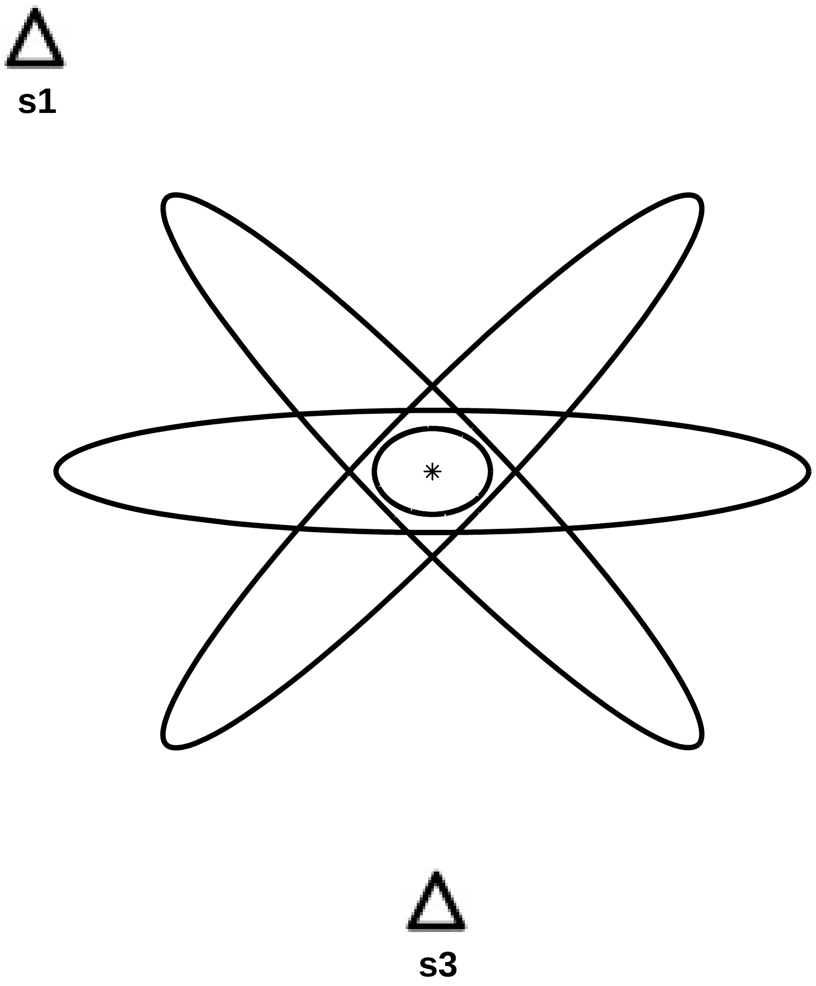
$$\mathbf{z}_k^c = \mathbf{t}[\mathbf{z}_k^p] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **Cartesian error covariance (time dependent):**

$$\boxed{\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\, \mathbf{R}\, \mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top}$$

- **sensor fusion: sensor-to-target-geometry enters into** $\mathbf{T}\mathbf{R}\mathbf{T}^\top$



Multiple sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^{\top}$.

Typical of radar, sonar, laser scanner (lidar), cameras, microphones, ...

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'Kalman gain matrix'}$$

Exercise 5.5

Do the same as in exercise 5.4, but use sensors that are producing range and azimuth measurements of the target positions.

Filtering Step: An Alternative Formulation

$$p(\mathbf{x}_k|\,\mathcal{Z}^k) = p(\mathbf{x}_k|\,\mathbf{z}_k, \mathcal{Z}^{k-1}) \qquad \text{(current measurement)}$$

$$= \frac{p(\mathbf{z}_k|\,\mathbf{x}_k)\; p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\; p(\mathbf{z}_k|\,\mathbf{x}_k)\; p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})} \qquad \text{(BAYES' rule)}$$

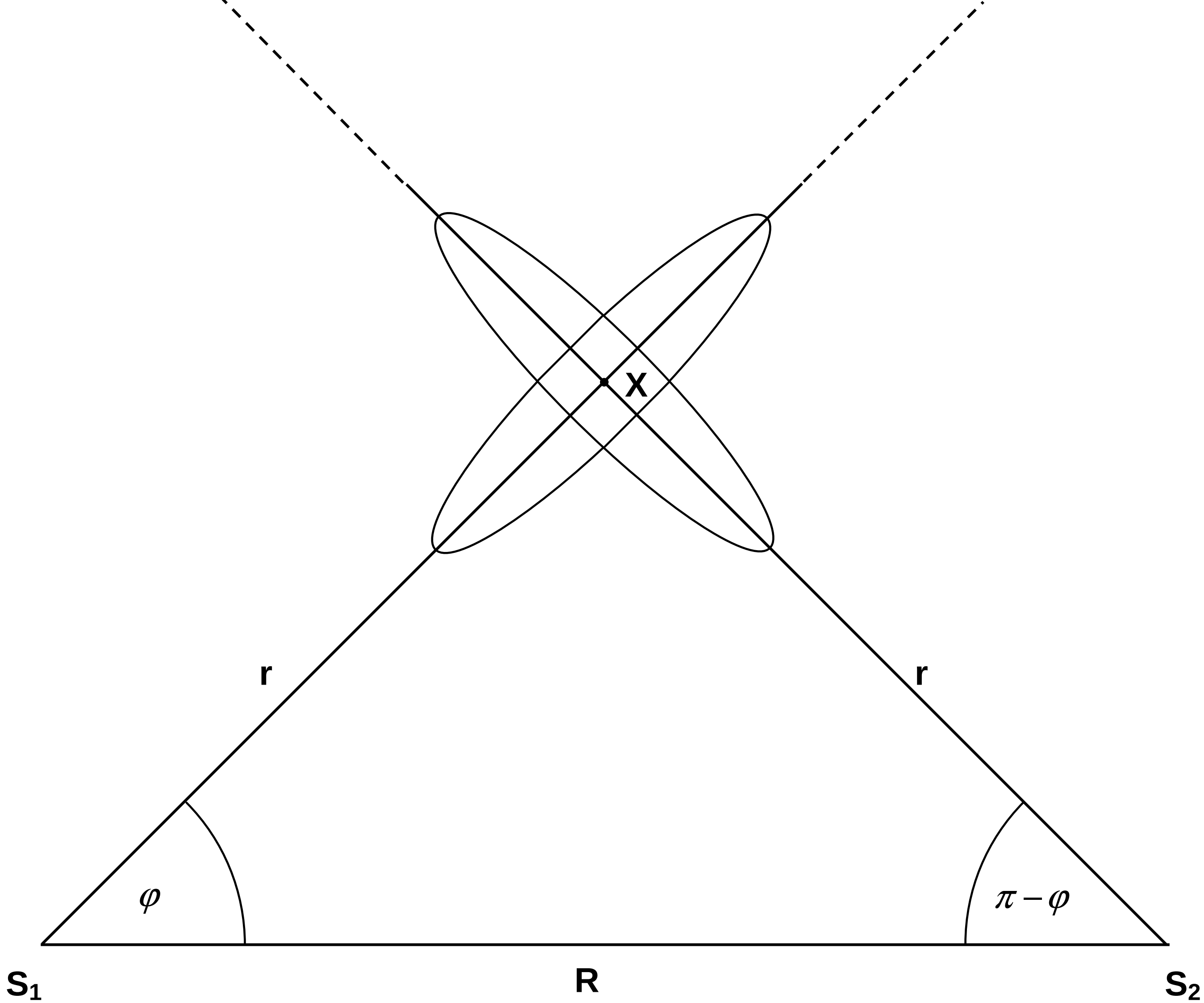
$$= \frac{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)\; \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}{\int d\mathbf{x}_k\; \underbrace{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)}_{\text{likelihood function}}\; \underbrace{\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}_{\text{prediction for } t_k}}$$

$$= \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}) \qquad \text{(product formula: 2. version!)}$$

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\big(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{z}_k\big)$$

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{H}$$

inverse covariance matrices are called information matrices.



X
r
r
φ
$\pi - \varphi$
S_1
R
S_2

Special case: *stationary* object

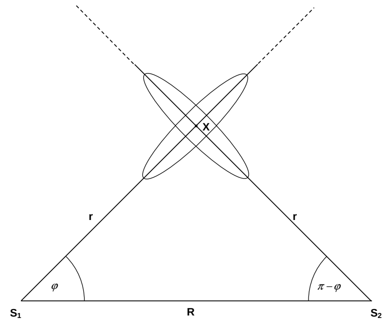
Example: different sensors $\mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$
$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k$ ***time dependent!***

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Filtering: $\mathbf{x}_{k|k} = \mathbf{P}_{k|k} \sum_{i=1}^{k} \mathbf{R}_i^{-1} \mathbf{z}_i, \qquad \mathbf{P}_{k|k} = \left(\sum_{i=1}^{k} \mathbf{R}_i^{-1} \right)^{-1}$

Kalman filter $\rightarrow$ *weighted*, recursive, arithmetic mean

estimation error covariance matrix: harmonic mean of measurement error matrices!



Discussion: stationary objects

- If all measurement error covariances R_i, $i = 1, \ldots, k$ are identical, we observe the statistical "square-root effect": $P_{k|k} = R/k$
- If the corresponding error ellipses are significantly different in their geometric extension, we can observe a much larger effect.
- statistical "intersection" of error ellipses: *harmonic mean*!
- In the limiting case of very eccentric error ellipses, we obtain triangulation of a position from bearings ($\rightarrow$ multiple sensor data fusion!).
- These considerations are valid also for 3D and more abstract measurements. The corresponding intersections: not intuitively clear.

Fraunhofer FKIE

Sensor data: range, azimuth, range-rate

Coordinates: Sensor data $\to$ *polar*, object evolution $\to$ *Cartesian*

Dynamics system: $\mathbf{x}^d = (x, y, \dot{x}, \dot{y})$

Sensor system: $\mathbf{x}^s = (r, \varphi, \dot{r}, \dot{\varphi})$

$$p\left(\mathbf{x}^d_{k-1} \middle| \mathcal{Z}^{k-1}\right) \xrightarrow[\text{Dynamics}]{p\left(\mathbf{x}^d_k \middle| \mathbf{x}^d_{k-1}\right)} p\left(\mathbf{x}^d_k \middle| \mathcal{Z}^{k-1}\right) \qquad p\left(\mathbf{x}^d_k \middle| \mathcal{Z}^k\right)$$

$$\uparrow \mathbf{t}_{d\leftarrow s} \qquad \downarrow \mathbf{t}_{s\leftarrow d} \qquad \uparrow \mathbf{t}_{d\leftarrow s}$$

$$p\left(\mathbf{x}^s_{k-1} \middle| \mathcal{Z}^{k-1}\right) \qquad p\left(\mathbf{x}^s_k \middle| \mathcal{Z}^{k-1}\right) \xrightarrow[\text{Sensor}]{p\left(\mathbf{z}_k \middle| \mathbf{x}^s_k\right)} p\left(\mathbf{x}^s_k \middle| \mathcal{Z}^k\right)$$

scan $k-1$ | scan k | scan k

non-linear **coordinate transformations:**

$$\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s] = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{pmatrix} \qquad \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d] = \begin{pmatrix} r \\ \varphi \\ \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan y/x \\ (x\dot{x}+y\dot{y})/\sqrt{x^2+y^2} \\ (x\dot{y}-y\dot{x})/(x^2+y^2) \end{pmatrix}$$

Extended *Kalman* filter: be wise - linearize!

non-lineare transformations: Taylor expansion up to 1st order

around $\mathbf{x}^s_{k|k}$ (filtering): $\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_k] \approx \mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]\,(\mathbf{x}^s_k - \mathbf{x}^s_{k|k})$

mit: $\mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] = \partial \mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] / \partial \mathbf{x}^s_{k|k}$ (Jacobian)

around $\mathbf{x}^d_{k|k-1}$ (Prediction): $\mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_k] \approx \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_{k|k-1}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}]\,(\mathbf{x}^d_k - \mathbf{x}^d_{k|k-1})$

with: $\mathbf{T}_{s\leftarrow d} = \partial \mathbf{t}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}] / \partial \mathbf{x}^d_{k|k-1}$

affine transformation of Gaussian random variables:

$$\mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{X}\big) \xrightarrow{y = \mathbf{a} + \mathbf{A}x} \mathcal{N}\big(y;\, \mathbf{a} + \mathbf{A}\mathbf{x},\, \mathbf{A}\mathbf{X}\mathbf{A}^\top\big)$$

Exercise 5.6 (voluntary!) Calculate Jacobians $\mathbf{T}_{d\leftarrow s}$ and $\mathbf{T}_{s\leftarrow d}$.

Recapitulation: How to deal with pdfs?

Example: Consider a RV with pdf $p(\mathbf{x})$! How to calculate the pdf $p(\mathbf{y})$ of a RV $\mathbf{y} = \mathbf{t}[\mathbf{x}]$ resulting from $\mathbf{x}$ by a transformation $\mathbf{t} : \mathbf{x} \mapsto \mathbf{t}[\mathbf{x}]$?

Recapitulation: How to deal with pdfs?

Example: Consider a RV with pdf $p(\mathbf{x})$! How to calculate the pdf $p(\mathbf{y})$ of a RV $\mathbf{y} = \mathbf{t}[\mathbf{x}]$ resulting from $\mathbf{x}$ by a transformation $\mathbf{t} : \mathbf{x} \mapsto \mathbf{t}[\mathbf{x}]$?

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x}, \mathbf{y}) \qquad \text{marginalize: bring } \mathbf{x} \text{ into the play!}$$

$$= \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) \qquad \text{notion of a conditional pdf}$$

Recapitulation: How to deal with pdfs?

Example: Consider a RV with pdf $p(\mathbf{x})$! How to calculate the pdf $p(\mathbf{y})$ of a RV $\mathbf{y} = \mathbf{t}[\mathbf{x}]$ resulting from $\mathbf{x}$ by a transformation $\mathbf{t} : \mathbf{x} \mapsto \mathbf{t}[\mathbf{x}]$?

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$$= \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) \qquad \text{notion of a conditional pdf}$$

$$= \int d\mathbf{x}\, \delta(\mathbf{y} - \mathbf{t}[\mathbf{x}])\, p(\mathbf{x}) \qquad \text{certain knowledge, if } \mathbf{x} \text{ is known!}$$

For Dirac-distributions: “integration formula”: $\int dx\, \delta(x - y)\, f(x) = f(y)$.

Assumption: invertable transformation $\mathbf{t} : \mathbf{x} \mapsto \mathbf{z} = \mathbf{t}[\mathbf{x}]$! Substitute in the integral: $\mathbf{x} = \mathbf{t}^{-1}[\mathbf{z}]$.

Remember: substitution rule for volume integrals

$$\varphi : y \mapsto \varphi[y] = x \qquad \int_a^b dx\, f(x) = \int_{\varphi[a]}^{\varphi[b]} dy\, \frac{d\varphi[y]}{dy}\, f(\varphi[y])$$

$$\boldsymbol{\varphi} : \mathbf{y} \mapsto \boldsymbol{\varphi}[\mathbf{y}] = \mathbf{x} \qquad \int_X d\mathbf{x}\, \mathbf{f}(\mathbf{x}) = \int_{\boldsymbol{\varphi}[X]} d\mathbf{y}\, \left| \frac{\partial \boldsymbol{\varphi}[\mathbf{y}]}{\partial \mathbf{y}} \right| \mathbf{f}(\boldsymbol{\varphi}[\mathbf{y}])$$

Jacobian $=$ matrix of the first derivatives of a vector-variate function

$\boldsymbol{\varphi} : \mathbf{x} \mapsto \boldsymbol{\varphi}[\mathbf{x}] = (\varphi_1[\mathbf{x}], \ldots, \varphi_m[\mathbf{x}])^\top$**,** $\mathbf{x} = (x_1, \ldots, x_n)^\top$**:**

$$\boldsymbol{\Phi} = \begin{pmatrix} \frac{\partial \varphi_1[\mathbf{x}]}{\partial x_1} & \ldots & \frac{\partial \varphi_m[\mathbf{x}]}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_1[\mathbf{x}]}{\partial x_n} & \ldots & \frac{\partial \varphi_m[\mathbf{x}]}{\partial x_n} \end{pmatrix} =: \frac{\partial \boldsymbol{\varphi}[\mathbf{x}]}{\partial \mathbf{x}}$$

Recapitulation: How to deal with pdfs?

Example: Consider a RV with pdf $p(\mathbf{x})$! How to calculate the pdf $p(\mathbf{y})$ of a RV $\mathbf{y} = \mathbf{t}[\mathbf{x}]$ resulting from $\mathbf{x}$ by a transformation $\mathbf{t} : \mathbf{x} \mapsto \mathbf{t}[\mathbf{x}]$?

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x}, \mathbf{y}) \qquad \text{marginalize: bring } \mathbf{x} \text{ into the play!}$$

$$= \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) \qquad \text{notion of a conditional pdf}$$

$$= \int d\mathbf{x}\, \delta(\mathbf{y} - \mathbf{t}[\mathbf{x}])\, p(\mathbf{x}) \qquad \text{certain knowledge, if } \mathbf{x} \text{ is known!}$$

For Dirac-distributions: “integration formula”: $\int dx\, \delta(x - y)\, f(x) = f(y)$.

Assumption: invertable transformation $\mathbf{t} : \mathbf{x} \mapsto \mathbf{z} = \mathbf{t}[\mathbf{x}]$! Substitute in the integral: $\mathbf{x} = \mathbf{t}^{-1}[\mathbf{z}]$.

Corresponding Jacobi determinant: $|\mathbf{T}^{-1}(\mathbf{z})|$ mit $\mathbf{T}^{-1}(\mathbf{z})| = \frac{\partial \mathbf{t}^{-1}[\mathbf{z}]}{\partial \mathbf{z}}$.

$$p(\mathbf{y}) = \int d\mathbf{z}\, |\mathbf{T}^{-1}(\mathbf{z})|\, \delta(\mathbf{y} - \mathbf{z})\, p(\mathbf{t}^{-1}[\mathbf{z}])$$

$$= |\mathbf{T}^{-1}(\mathbf{y})|\, p(\mathbf{t}^{-1}[\mathbf{y}]) =: \mathcal{T}[p](\mathbf{y}) \qquad \mathcal{T} \text{ is called “Transfer-Operator”}$$

To be generalized under certain assumptions!

Let $\mathbf{x}$ be a Gaussian RV with $p(\mathbf{x}) = \mathcal{N}\big(\mathbf{x};\, \mathbb{E}[\mathbf{x}],\, \mathbb{C}[\mathbf{x}]\big)$. Show for the pdf of the RV $\mathbf{y} = \mathbf{t}[\mathbf{x}]$, resulting from $\mathbf{x}$ by an *affine* transformation $\mathbf{t}[\mathbf{x}] = \mathbf{a} + \mathbf{A}\mathbf{x}$:

Exercise 5.7 (voluntary)

$$p(\mathbf{y}) = \left|\frac{\partial \mathbf{t}^{-1}[\mathbf{y}]}{\partial \mathbf{y}}\right| p\big(\mathbf{t}^{-1}[\mathbf{y}]\big)$$
$$= \mathcal{N}\big(\mathbf{y};\, \mathbf{a} + \mathbf{A}\mathbb{E}[\mathbf{x}],\, \mathbf{A}\mathbb{E}[\mathbf{x}]\mathbf{A}^\top\big)$$

$\mathbf{a}$, $\mathbf{A}$: vector/matrix of suitable dimension (constant),

$$\mathbf{t}^{-1} : \mathbf{y} \mapsto \mathbf{t}^{-1}[\mathbf{y}] = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{a}), \quad \frac{\partial \mathbf{t}^{-1}[\mathbf{y}]}{\partial \mathbf{y}} = \mathbf{A}^{-1}$$

Remember some rules for dealing with matrices:

$$|\mathbf{A}||\mathbf{B}| = |\mathbf{AB}|,\ |A^{-1}| = |\mathbf{A}|^{-1}$$

$$(\mathbf{AB})^\top = \mathbf{B}^\top\mathbf{A}^\top,\ (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$