Multiple Hypothesis Tracking: Basic Idea

Iterative updating of conditional probability densities!

kinematic target state x_k at time t_k, **accumulated sensor data** $\mathcal{Z}^k$

a priori knowledge: target dynamics models, sensor model, other context

- **prediction:** $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \xrightarrow[\text{context}]{\text{dynamics model}} p(\mathbf{x}_k|\mathcal{Z}^{k-1})$
- **filtering:** $p(\mathrm{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor model}]{\text{sensor data } Z_k} p(\mathrm{x}_k|\mathcal{Z}^k)$
- **retrodiction:** $p(\mathbf{x}_{l-1}|\mathcal{Z}^k) \xleftarrow[\text{dynamics model}]{\text{filtering output}} p(\mathbf{x}_l|\mathcal{Z}^k)$

– ***finite mixture:*** **inherent ambiguity (data, model, road *network*)**
– ***optimal estimators:*** **e.g. minimum mean squared error (MMSE)**
– ***initiation of pdf iteration:*** **multiple hypothesis track extraction**

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\ \mathbf{x}_{0|0},\ \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}_{k-1|k-1},\ \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\ \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\ \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k},\ \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$

$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$ 'KALMAN gain matrix'

Stationary object position from noisy measurements

position: $x \in \mathbb{R}$, measurements: z_i with "likelihood": $p(z_i|x) = \mathcal{N}(z_i; x, \sigma_i)$

each measurement can have its own measurement error σ_i (standard deviation)

Initial knowledge: $p(x) = \mathcal{N}(x; x_0, \Sigma_0), \quad \Sigma_0 \gg \sigma_1,$ flat.

Impact of the first measurement: $p(x|z_1) = \mathcal{N}(x; z_1, \sigma_1)$

$$p(x|z_1) = p(z_1|x)\, p(x) \,/\, \int dx\, p(z_1|x)\, p(x)$$ Bayes

$$p(x|z_1) = p(z_1|x)\, p(x) \;/\; \int dx\, p(z_1|x)\, p(x) \qquad \text{Bayes}$$

$$\propto \mathcal{N}(z_1; x, \sigma_1)\, \mathcal{N}(x; x_0, \Sigma_0) \qquad \text{up to normalization}$$

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$$\propto \exp\left\{-\tfrac{1}{2}\left((x - x_0)^2/\Sigma_0^2 + (z_1 - x)^2/\sigma_1^2\right)\right\} \qquad \text{Ignore constants!}$$

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p(x|z_1) &= p(z_1|x)\, p(x) \;/ \int dx\, p(z_1|x)\, p(x) && \text{Bayes} \\
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&= \exp\left\{-\tfrac{1}{2}\left(x^2 - 2xx_1 + x_1^2 - x_1^2\right)/\Sigma_1^2\right\} && \text{Add Zero!}\\
&\propto \exp\left\{-\tfrac{1}{2}(x-x_1)^2/\Sigma_1^2\right\}\\
&\propto \mathcal{N}(x; x_1, \Sigma_1)
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$$

$$
\begin{aligned}
\text{with:}\quad \Sigma_1 &= \sqrt{1/(1/\Sigma_0^2 + 1/\sigma_1^2)} && \to \sigma_1 \quad (\Sigma_0 \gg \sigma_1)\\
x_1 &= \Sigma_1^2(x_0/\Sigma_0^2 + z_1/\sigma_1^2) && \to z_1
\end{aligned}
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Stationary object position from noisy measurements

position: $x \in \mathbb{R}$, measurements: z_i with “likelihood”: $p(z_i|x) = \mathcal{N}(z_i; x, \sigma_i)$

each measurement can have its own measurement error σ_i (standard deviation)

Initial knowledge: $p(x) = \mathcal{N}(x; x_0, \Sigma_0), \quad \Sigma_0 \gg \sigma_1,$ flat.

Impact of the first measurement: $p(x|z_1) = \mathcal{N}(x; z_1, \sigma_1)$

Impact of k measurement: $p(x|z_k, \ldots, z_1) = \mathcal{N}(x; x_k, \Sigma_k)$

with: $1/\Sigma_k^2 = \sum_{i=1}^{k} 1/\sigma_i^2$ harmonic mean, $x_k = \Sigma_k^2 \left(\sum_{i=1}^{k} z_i/\sigma_i^2 \right)$

$$p(x|z_k, \ldots, k_1) \propto p(z_k|x)\, p(x|z_{k-1}, \ldots, z_1) \qquad \text{Bayes}$$

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p(x|z_k,\dots,k_1) &\propto p(z_k|x)\,p(x|z_{k-1},\dots,z_1) && \text{Bayes}\\
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$$\propto \exp\left\{-\tfrac{1}{2}\left((x^2 - 2xx_{k-1})/\Sigma_{k-1}^2 + (-2z_k x + x^2)/\sigma_k^2\right)\right\}$$

$$= \exp\left\{-\tfrac{1}{2}\left(x^2(1/\Sigma_{k-1}^2 + 1/\sigma_k^2) - 2x(x_{k-1}/\Sigma_{k-1}^2 + z_k/\sigma_k^2)\right)\right\}$$

$$= \exp\left\{-\tfrac{1}{2}\left(x^2/\Sigma_k^2 - 2x(x_{k-1}/\Sigma_{k-1}^2 + z_k/\sigma_k^2)\right)\right\}, \quad \Sigma_k^{-2} = \Sigma_{k-1}^{-2} + \sigma_k^{-2}$$

$$= \exp\left\{-\tfrac{1}{2}\left(x^2 - 2xx_k\right)/\Sigma_k^2\right\}, \qquad x_k = (x_{k-1}/\Sigma_{k-1}^2 + z_k/\sigma_k^2)\Sigma_k^2$$

$$= \exp\left\{-\tfrac{1}{2}\left(x^2 - 2xx_k + x_k^2 - x_k^2\right)/\Sigma_k^2\right\}$$

$$
\begin{aligned}
p(x|z_k, \ldots, k_1) &\propto p(z_k|x)\, p(x|z_{k-1}, \ldots, z_1) && \text{Bayes} \\
&\propto \mathcal{N}(z_k; x, \sigma_k)\, \mathcal{N}(x; x_{k-1}, \Sigma_{k-1}) && \text{induction assumption} \\
&\propto \text{exp}\left\{-\tfrac{1}{2}\left((x - x_{k-1})^2/\Sigma^2_{k-1} + (z_k - x)^2/\sigma^2_k\right)\right\} \\
&\propto \text{exp}\left\{-\tfrac{1}{2}\left((x^2 - 2xx_{k-1})/\Sigma^2_{k-1} + (-2z_k x + x^2)/\sigma^2_k\right)\right\} \\
&= \text{exp}\left\{-\tfrac{1}{2}\left(x^2(1/\Sigma^2_{k-1} + 1/\sigma^2_k) - 2x(x_{k-1}/\Sigma^2_{k-1} + z_k/\sigma^2_k)\right)\right\} \\
&= \text{exp}\left\{-\tfrac{1}{2}\left(x^2/\Sigma^2_k - 2x(x_{k-1}/\Sigma^2_{k-1} + z_k/\sigma^2_k)\right)\right\}, && \Sigma_k^{-2} = \Sigma_{k-1}^{-2} + \sigma_k^{-2} \\
&= \text{exp}\left\{-\tfrac{1}{2}\left(x^2 - 2xx_k\right)/\Sigma^2_k\right\}, && x_k = (x_{k-1}/\Sigma^2_{k-1} + z_k/\sigma^2_k)\Sigma^2_k \\
&= \text{exp}\left\{-\tfrac{1}{2}\left(x^2 - 2xx_k + x_k^2 - x_k^2\right)/\Sigma^2_k\right\} \\
&\propto \text{exp}\left\{-\tfrac{1}{2}\left(x - x_k\right)^2/\Sigma^2_k\right\}
\end{aligned}
$$

$$p(x|z_k, \ldots, k_1) \propto p(z_k|x)\, p(x|z_{k-1}, \ldots, z_1) \qquad \text{Bayes}$$

$$\propto \mathcal{N}(z_k; x, \sigma_k)\, \mathcal{N}(x; x_{k-1}, \Sigma_{k-1}) \qquad \text{induction assumption}$$

$$\propto \exp\left\{-\tfrac{1}{2}\left((x - x_{k-1})^2/\Sigma_{k-1}^2 + (z_k - x)^2/\sigma_k^2\right)\right\}$$

$$\propto \exp\left\{-\tfrac{1}{2}\left((x^2 - 2xx_{k-1})/\Sigma_{k-1}^2 + (-2z_k x + x^2)/\sigma_k^2\right)\right\}$$

$$= \exp\left\{-\tfrac{1}{2}\left(x^2(1/\Sigma_{k-1}^2 + 1/\sigma_k^2) - 2x(x_{k-1}/\Sigma_{k-1}^2 + z_k/\sigma_k^2)\right)\right\}$$

$$= \exp\left\{-\tfrac{1}{2}\left(x^2/\Sigma_k^2 - 2x(x_{k-1}/\Sigma_{k-1}^2 + z_k/\sigma_k^2)\right)\right\}, \quad \Sigma_k^{-2} = \Sigma_{k-1}^{-2} + \sigma_k^{-2}$$

$$= \exp\left\{-\tfrac{1}{2}\left(x^2 - 2xx_k\right)/\Sigma_k^2\right\}, \qquad x_k = (x_{k-1}/\Sigma_{k-1}^2 + z_k/\sigma_k^2)\Sigma_k^2$$

$$= \exp\left\{-\tfrac{1}{2}\left(x^2 - 2xx_k + x_k^2 - x_k^2\right)/\Sigma_k^2\right\}$$

$$\propto \exp\left\{-\tfrac{1}{2}(x - x_k)^2/\Sigma_k^2\right\}$$

$$\propto \mathcal{N}(x; x_k, \Sigma_k)$$

with: $$\Sigma_k = 1/\sqrt{\sum_{i=1}^{k} 1/\sigma_i^2} \qquad \text{harmonic mean}$$

$$x_k = \Sigma_k^2 \left(\sum_{i=1}^{k} z_i/\sigma_i^2\right) \qquad \text{weighted arithmetic mean}$$

Stationary object position from noisy measurements

position: $x \in \mathbb{R}$, measurements: z_i with "likelihood": $p(z_i|x) = \mathcal{N}(z_i; x, \sigma_i)$

each measurement can have its own measurement error σ_i (standard deviation)

Initial knowledge: $p(x) = \mathcal{N}(x; x_0, \Sigma_0), \quad \Sigma_0 \gg \sigma_1,$ flat.

Impact of the first measurement: $p(x|z_1) = \mathcal{N}(x; z_1, \sigma_1)$

Impact of k measurement: $p(x|z_k, \ldots, z_1) = \mathcal{N}(x; x_k, \Sigma_k)$

with: $1/\Sigma_k^2 = \sum_{i=1}^{k} 1/\sigma_i^2$ harmonic mean, $x_k = \Sigma_k^2 \left(\sum_{i=1}^{k} z_i/\sigma_i^2 \right)$

For $\sigma = \sigma_i \; \forall i$: $\Sigma_k = \sigma/\sqrt{k}$ "square-root law"

$$x_k = \frac{1}{k} \sum_{i=1}^{k} z_i$$ arithmetic mean

Fraunhofer FKIE

The Multivariate GAUSSian Pdf

– *wanted:* probabilities ‘concentrated’ around a center $\mathbf{x}$

– *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

$q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

– *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\, \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top\mathbf{P}^{-1}(x-\mathbf{x})}$$

Exercise 4.1 Show: $\int dx\, \mathrm{e}^{-q(x)} = \sqrt{|2\pi\mathbf{P}|}$, $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P}$

Trick: Symmetric, positively definite matrices can be diagonalized by an orthogonal coordinate transform.

The Multivariate GAUSSian Pdf

– *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$

– *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

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$$\mathbb{E}[x] = \mathbf{x},\ \ \mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P} \quad \text{(covariance)}$$

The Multivariate GAUSSian Pdf

– *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$

– *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

$q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

– *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\, \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top \mathbf{P}^{-1}(x-\mathbf{x})}$$

$$\mathbb{E}[x] = \mathbf{x}, \;\; \mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P} \;\; \text{(covariance)}$$

– GAUSS*ian Mixtures:* $p(x) = \sum_i p_i\, \mathcal{N}(x;\, \mathbf{x}_i,\, \mathbf{P}_i)$ (weighted sums)

A Useful Product Formula for Gaussians

$$\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\, \mathbf{y},\, \mathbf{P}\big) = \underbrace{\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S}\big)}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}\big(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top\big) \\ \mathcal{N}\big(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}\big) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Sketch of the proof:

- Interpret $\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\, \mathbf{y},\, \mathbf{P}\big)$ as a joint pdf $p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{z}, \mathbf{x})$.
- Show that $p(\mathbf{z}, \mathbf{x})$ is a Gaussian: $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\Big(\big(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\big);\, \big(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\big),\, \big(\begin{smallmatrix}\mathbf{S} & \mathbf{H}\mathbf{P}\\ \mathbf{P}\mathbf{H}^\top & \mathbf{P}\end{smallmatrix}\big)\Big)$.
- Calculate from $p(\mathbf{z}, \mathbf{x})$ the marginal and conditional pdfs $p(\mathbf{z})$ and $p(\mathbf{x}|\mathbf{z})$.
- From $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) = p(\mathbf{x}, \mathbf{z})$ we obtain the result.

Fraunhofer FKIE

Affine Transforms of Gaussian Random Variables

$$\mathcal{N}\big(\mathbf{x};\, \mathbb{E}[\mathbf{x}],\, \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\, \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\, \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\Big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\Big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\Big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\Big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

Affine Transforms of Gaussian Random Variables

$$\boxed{\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)}$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{D}\big)$ with $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D} \to 0$$

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{D}\big)$ with $\mathbf{D}\to\mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D}\to 0$$

$$= \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top+\mathbf{D}\big) \qquad \text{for } \mathbf{D}\to 0; \quad \text{product formula!}$$

Also true if $\mathrm{dim}(\mathbf{x}) \neq \mathrm{dim}(\mathbf{y})$**!**

Fraunhofer FKIE

A popular model for object evolutions

Piecewise Constant White Acceleration Model

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

For known $\mathbf{x}_{k-1}$ and without external influences we have with $\Delta T_k = t_k - t_{k-1}$:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{I} & \Delta T_k \, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1} \mathbf{x}_{k-1},$$

see blackboard!

Assume during the interval ΔT_k a constant acceleration $\mathbf{a}_k$ causing the state evolution:

$$\begin{pmatrix} \frac{1}{2} \Delta T_k^2 \, \mathbf{I} \\ \Delta T_k \, \mathbf{I} \end{pmatrix} \mathbf{a}_k =: \mathbf{G}_k \mathbf{a}_k,$$

linear transform!

Let $\mathbf{a}_k$ be a Gaussian RV with pdf: $p(\mathbf{a}_k) = \mathcal{N}\big(\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2 \mathbf{I}\big)$, we therefore have:

$$p(\mathbf{G}_k \mathbf{a}_k) = \mathcal{N}\big(\mathbf{G}_k \mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2 \mathbf{G}_k \mathbf{G}_k^\top\big).$$

Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\, \mathbf{I} & \frac{1}{2}\Delta T_k^3\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \Delta T_k^2\, \mathbf{I} \end{pmatrix}$$

Exercise 4.2 Consider $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$ (position, velocity, acceleration)

Show that $\mathbf{F}_{k|k-1}$ and $\mathbf{D}_{k|k-1} = \Sigma_k^2 \mathbf{G}_k \mathbf{G}_k^\top$ (constant acceleration rates) are given by:

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\, \mathbf{I} & \frac{1}{2}\Delta T_k^2\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\, \mathbf{I} & \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \frac{1}{2}\Delta T_k^2\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \Delta T_k^2 \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^2\, \mathbf{I} & \Delta T_k \mathbf{I} & \mathbf{I} \end{pmatrix}$$

with $\Delta T_k = t_k - t_{k-1}$. Reasonable choice: $\frac{1}{2} q_{\text{max}} \leq \Sigma_k \leq q_{\text{max}}$

Sensor Fusion: Gain in Localization Accuracy

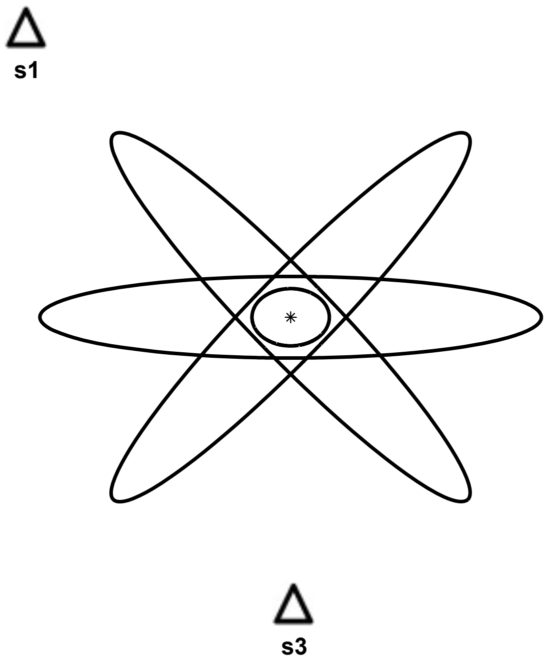
If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

Fraunhofer FKIE

Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

a closer look: The error of each measurement $\mathbf{z}_i$ is described by a related measurement error *covariance matrix* $\mathbf{R}_i$ ('error ellipsoids'). In 2 dimensions:



$\mathbf{R}_i$ can strongly depend on the underlying senor-to-target geometry!

Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

  $\mathbf{z}_k = (r_k, \varphi_k)^\top$, measurement error: $\mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}$, r, φ independent

Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$\mathbf{z}_k = (r_k, \varphi_k)^\top$, measurement error: $\mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}$, r, φ independent

Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

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Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \ r, \varphi \text{ independent}$$

Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

- **What is the likelihood function in Cartesian coordinates?**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix}$$

Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

constant and linear term of a Taylor series only, blackboard!

Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \quad r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \quad r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **affine transform of GAUSSian random variables:**

$$\mathcal{N}\big(\mathbf{z};\ \mathbf{x},\ \mathbf{R}\big) \xrightarrow{\mathbf{z}'=\mathbf{t}+\mathbf{T}\mathbf{z}} \mathcal{N}\big(\mathbf{z}';\ \mathbf{t}+\mathbf{T}\mathbf{x},\ \mathbf{T}\mathbf{R}\mathbf{T}^\top\big)$$

Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \quad r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

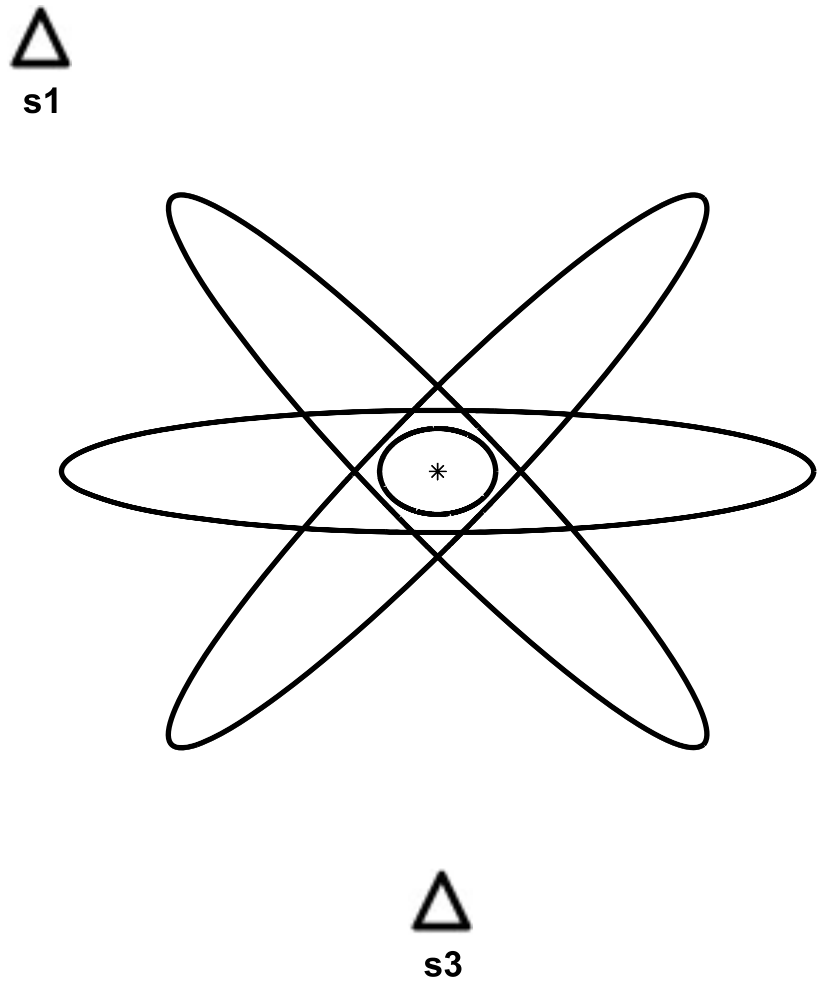
$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **Cartesian error covariance (time dependent):**

$$\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\, \mathbf{R}\, \mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top$$

- **sensor fusion: sensor-to-target-geometry enters into $\mathbf{T}\mathbf{R}\mathbf{T}^\top$**



sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$

Filtering step: alternative formulation

$$p\big(\mathbf{x}_k|\,\mathcal{Z}^k\big) = p\big(\mathbf{x}_k|\,\mathbf{z}_k, \mathcal{Z}^{k-1}\big) \qquad \text{(current measurement)}$$

$$= \frac{p\big(\mathbf{z}_k|\,\mathbf{x}_k\big)\, p\big(\mathbf{x}_k|\,\mathcal{Z}^{k-1}\big)}{\int d\mathbf{x}_k\, p\big(\mathbf{z}_k|\,\mathbf{x}_k\big)\, p\big(\mathbf{x}_k|\,\mathcal{Z}^{k-1}\big)} \qquad \text{(Bayes' rule)}$$

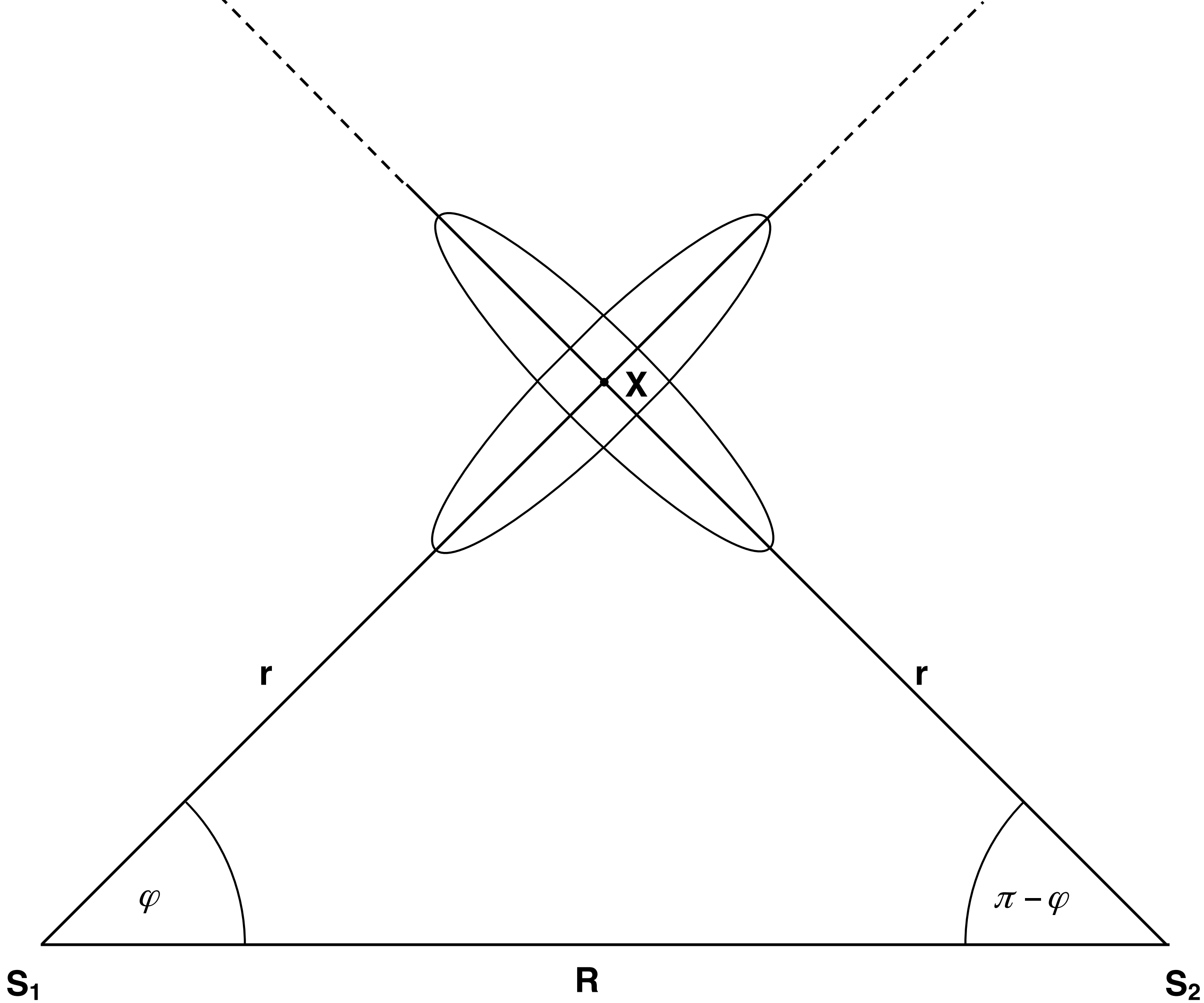
$$= \frac{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}{\int d\mathbf{x}_k\, \underbrace{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)}_{\text{likelihood function}}\, \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}_{\text{prediction for } t_k}}$$

$$= \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big) \qquad \text{(product formula: 2. version!)}$$

$$\begin{aligned} \mathbf{x}_{k|k} &= \mathbf{P}_{k|k}^{-1}\big(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{z}_k\big) \\ \mathbf{P}_{k|k}^{-1} &= \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top\mathbf{R}_k^{-1}\mathbf{H} \end{aligned}$$

inverse covariance matrices are called information matrices.

Fraunhofer FKIE


X
r
r
φ
π − φ
S1
R
S2

Special case: *stationary* object

Example: different sensors $\mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$
$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k$ ***time dependent!***

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Prediction: $\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}, \quad \mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}^\top + \mathbf{D}_{k|k-1}$

Filtering: $\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\big(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{z}_k\big)$ (2. formulation)

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{H}$$

Special case: *stationary* object

Example: different sensors

$$\mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$$
$$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k \quad \textit{time dependent!}$$

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Prediction: $\mathbf{x}_{k|k-1} = \mathbf{x}_{k-1|k-1}, \quad \mathbf{P}_{k|k-1} = \mathbf{P}_{k-1|k-1}$

Filtering:

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\left(\mathbf{P}_{k-1|k-1}^{-1}\mathbf{x}_{k-1|k-1} + \mathbf{R}_k^{-1}\mathbf{z}_k\right)$$

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k-1|k-1}^{-1} + \mathbf{R}_k^{-1}$$

Special case: *stationary* object

Example: different sensors $\mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$
$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k$ ***time dependent!***

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Prediction: $\mathbf{x}_{k|k-1} = \mathbf{x}_{k-1|k-1}, \quad \mathbf{P}_{k|k-1} = \mathbf{P}_{k-1|k-1}$

Filtering:

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\big(\mathbf{P}_{k-1|k-1}^{-1}\mathbf{x}_{k-1|k-1} + \mathbf{R}_k^{-1}\mathbf{z}_k\big) = \mathbf{P}_{k|k}^{-1}\sum_{i=1}^{k}\mathbf{R}_i^{-1}\mathbf{z}_i$$

$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k-1|k-1}^{-1} + \mathbf{R}_k^{-1} = \sum_{i=1}^{k}\mathbf{R}_i^{-1}$$

Special case: *stationary* object

Example: different sensors

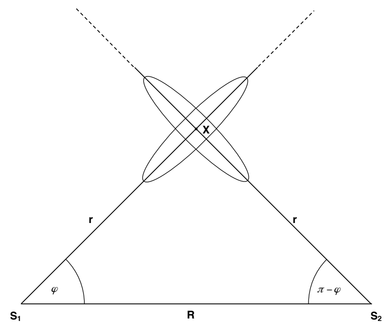
$$\mathbf{F} = \mathbf{I} \quad \mathbf{D} = \mathbf{O}$$
$$\mathbf{H} = \mathbf{I} \quad \mathbf{R}_k \quad \textit{time dependent!}$$

Initiation: $\mathbf{x}_{1|1} = \mathbf{z}_1, \qquad \mathbf{P}_{1|1} = \mathbf{R}_1$

Filtering: $\mathbf{x}_{k|k} = \mathbf{P}_{k|k} \sum_{i=1}^{k} \mathbf{R}_i^{-1} \mathbf{z}_i, \qquad \mathbf{P}_{k|k} = \left(\sum_{i=1}^{k} \mathbf{R}_i^{-1} \right)^{-1}$

Kalman filter $\rightarrow$ *weighted*, recursive, arithmetic mean

estimation error covariance matrix: harmonic mean of measurement error matrices!



Fraunhofer FKIE

Discussion: stationary objects

- If all measurement error covariances R_i, $i = 1, \ldots, k$ are identical, we observe the statistical "square-root effect": $P_{k|k} = R/k$

- If the corresponding error ellipses are significantly different in their geometric extension, we can observe a much larger effect.

- statistical "intersection" of error ellipses: *harmonic mean*!

- In the limiting case of very eccentric error ellipses, we obtain triangulation of a position from bearings ($\rightarrow$ multiple sensor data fusion!).

- These considerations are valid also for 3D and more abstract measurements. The corresponding intersections: not intuitively clear.