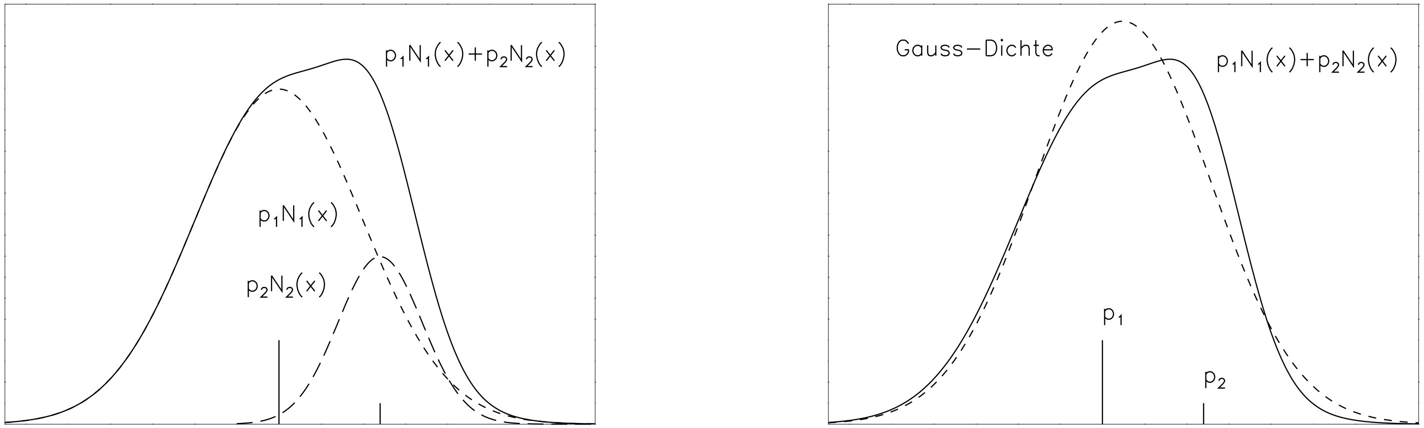
Moment Matching: Approximate an arbitrary pdf

$p(x)$ with $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{C}[x] = \mathbf{P}$ by $p(x) \approx \mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{P}\big)$!

here especially: $p(x) = \sum_i p_i \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i)$ (GAUSSian mixtures)



Exercise 7.1 Show:

$$\mathbf{x} = \sum_i p_i \, \mathbf{x}_i$$

$$\mathbf{P} = \sum_i p_i \left\{ \mathbf{P}_i + \overbrace{(\mathbf{x}_i - \mathbf{x})(\mathbf{x}_i - \mathbf{x})^\top}^{\text{spread term}} \right\}$$

Fraunhofer FKIE

2nd Order Approximation:

here: $p(x) = \sum_i p_H \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i) \approx \mathcal{N}(x; \mathbb{E}_p[x], \mathbb{C}_p[x])$

$$\mathbb{E}_p[x] = \int dx\, x\, p(x) = \sum_i p_i \int dx\, x\, \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i) = \sum_i p_i\, \mathbf{x}_i =: \mathbf{x}$$

$$\mathbb{C}_p[x] = \int dx\, p(x)\, (x - \mathbb{E}_p[x])(x - \mathbb{E}_p[x])^\top = \sum_i p_i \int dx\, (x - \mathbf{x})(x - \mathbf{x})^\top \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i)$$

$$= \sum_i p_i \int dx\, \{(x - \mathbf{x})(x - \mathbf{x})^\top - 2(x - \mathbf{x}_i)(\mathbf{x}_i - \mathbf{x})^\top\} \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i)$$

since we have: $\int dx\, (x - \mathbf{x}_i)(\mathbf{x}_i - \mathbf{x})^\top \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i) = 0$

$$= \sum_i p_i \int dx\, \{xx^\top - 2x\mathbf{x}_i^\top + \mathbf{x}_i\mathbf{x}_i^\top + \mathbf{x}_i\mathbf{x}_i^\top - 2\mathbf{x}_i\mathbf{x}^\top + \mathbf{x}\mathbf{x}^\top\} \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i)$$

$$= \sum_i p_i \int dx\, \{(x - \mathbf{x}_i)(x - \mathbf{x}_i)^\top + (\mathbf{x}_i - \mathbf{x})(\mathbf{x}_i - \mathbf{x})^\top\} \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i)$$

$$= \sum_i p_i \{\mathbf{P}_i + (\mathbf{x}_i - \mathbf{x})(\mathbf{x}_i - \mathbf{x})^\top\} = \mathbf{P}$$

Excursus: χ^2-distributed random variables

nD Gaussian RV $\mathbf{x}$ with expectation $\mathbb{E}[\mathbf{x}] = \bar{\mathbf{x}}$ and covariance $\mathbb{C}[\mathbf{x}] = \mathbf{P}$

What is the pdf of the derived scalar RV $\boxed{q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}})}$?

Excursus: χ^2-distributed random variables

nD Gaussian RV $\mathbf{x}$ with expectation $\mathbb{E}[\mathbf{x}] = \bar{\mathbf{x}}$ and covariance $\mathbb{C}[\mathbf{x}] = \mathbf{P}$

What is the pdf of the derived scalar RV $q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}})$?

affine transformation of Gaussian RVs:

$$\mathcal{N}\big(\mathbf{x};\, \bar{\mathbf{x}},\, \mathbf{X}\big) \xrightarrow{\mathbf{y}=\mathbf{a}+\mathbf{A}\mathbf{y}} \mathcal{N}\big(\mathbf{y};\, \mathbf{a} + \mathbf{A}\bar{\mathbf{x}},\, \mathbf{A}\mathbf{X}\mathbf{A}^\top\big)$$

Excursus: χ^2-distributed random variables

nD Gaussian RV $\mathbf{x}$ with expectation $\mathbb{E}[\mathbf{x}] = \bar{\mathbf{x}}$ and covariance $\mathbb{C}[\mathbf{x}] = \mathbf{P}$

What is the pdf of the derived scalar RV $\boxed{q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}})}$?

Consider transformed RV $\mathbf{u} = \mathbf{P}^{-1/2}(\mathbf{x} - \bar{\mathbf{x}})$! We have: $p(\mathbf{u}) = \mathcal{N}\big(\mathbf{u};\, \mathbf{o},\, \mathbf{I}\big)$

obviously: $q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) = \mathbf{u}^\top \mathbf{u} = \sum_{i=1}^{n} u_i^2$ with: $u_i \sim N(0, 1)$

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ with $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{O}, \mathbf{I})$. How is $x = \sum_{i=1}^{n} x_i^2$ distributed?

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ with $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{O}, \mathbf{I})$. How is $x = \sum_{i=1}^{n} x_i^2$ distributed?

$$p(x) = \int d\mathbf{x}\, p(x, \mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x})\, p(\mathbf{x})$$

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ with $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{O}, \mathbf{I})$. How is $x = \sum_{i=1}^{n} x_i^2$ distributed?

$$p(x) = \int d\mathbf{x}\, p(x, \mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x}) \prod_{i=1}^{n} p(x_i) \qquad \text{insert!}$$

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ with $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{O}, \mathbf{I})$. How is $x = \sum_{i=1}^{n} x_i^2$ distributed?

$$p(x) = \int d\mathbf{x}\, p(x, \mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x}) \prod_{i=1}^{n} p(x_i) \qquad \text{insert!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dx_1 \ldots \int dx_n\, \delta\Big(x - \textstyle\sum_{i=1}^{n} x_i^2\Big)\, \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} \qquad \text{polar coordinates!}$$

Fraunhofer FKIE

Polar coordinates in n dimensions:

$$
\begin{aligned}
x_1 &= r \cos\varphi_1 && (0 \le \varphi_1 \le \pi) \\
x_2 &= r \sin\varphi_1 \cos\varphi_2 && (0 \le \varphi_2 \le \pi) \\
&\dots \\
x_{n-1} &= r \sin\varphi_1 \dots \sin\varphi_{n-2} \cos\varphi_{n-1} && (0 \le \varphi_{n-1} < 2\pi) \\
x_n &= r \sin\varphi_1 \dots \sin\varphi_{n-1} && (0 \le r)
\end{aligned}
$$

We in particular have: $r^2 = x_1^2 + \dots + x_n^2$

Polar coordinates in n dimensions:

$$
\begin{aligned}
x_1 &= r\cos\varphi_1 && (0 \le \varphi_1 \le \pi)\\
x_2 &= r\sin\varphi_1\cos\varphi_2 && (0 \le \varphi_2 \le \pi)\\
\ldots\\
x_{n-1} &= r\sin\varphi_1\ldots\sin\varphi_{n-2}\cos\varphi_{n-1} && (0 \le \varphi_{n-1} < 2\pi)\\
x_n &= r\sin\varphi_1\ldots\sin\varphi_{n-1} && (0 \le r)
\end{aligned}
$$

We in particular have: $r^2 = x_1^2 + \ldots + x_n^2$

Substitution rule: $\displaystyle\int dx_1 \ldots \int dx_n\, f(x_1,\ldots,x_n) = \int dr \int d\Omega\; r^{n-1} f(r,\varphi_1,\ldots,\varphi_{n-1})$

Solid angle element: $d\Omega = \sin^{n-2}\varphi_1\ \sin^{n-3}\varphi_2 \ldots \sin^2\varphi_{n-3}\ \sin\varphi_{n-2}\, d\varphi_1 \ldots d\varphi_{n-1}$

Polar coordinates in n dimensions:

$$
\begin{aligned}
x_1 &= r \cos \varphi_1 & (0 \le \varphi_1 \le \pi) \\
x_2 &= r \sin \varphi_1 \cos \varphi_2 & (0 \le \varphi_2 \le \pi) \\
\ldots & & \\
x_{n-1} &= r \sin \varphi_1 \ldots \sin \varphi_{n-2} \cos \varphi_{n-1} & (0 \le \varphi_{n-1} < 2\pi) \\
x_n &= r \sin \varphi_1 \ldots \sin \varphi_{n-1} & (0 \le r)
\end{aligned}
$$

We in particular have: $r^2 = x_1^2 + \ldots + x_n^2$

Substitution rule: $\int dx_1 \ldots \int dx_n \, f(x_1, \ldots, x_n) = \int dr \int d\Omega \; r^{n-1} f(r, \varphi_1, \ldots, \varphi_{n-1})$

Solid angle element: $d\Omega = \sin^{n-2} \varphi_1 \; \sin^{n-3} \varphi_2 \ldots \sin^2 \varphi_{n-3} \; \sin \varphi_{n-2} \, d\varphi_1 \ldots d\varphi_{n-1}$

$f(\ldots) = 1$ **yields the volume of a nD-sphere with radius r:** $K_n(r) = \int_0^r dr \, r^{n-1} \int d\Omega = \frac{\pi^{\frac{n}{2}} r^n}{\frac{n}{2}\Gamma(\frac{n}{2})}$

Solid space angle integration: $r^{n-1} \int d\Omega = \frac{dK_n(r)}{dr} = \frac{\pi^{\frac{n}{2}}}{\frac{1}{2}\Gamma(\frac{n}{2})} \, r^{n-1}$

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ with $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{O}, \mathbf{I})$. How is $x = \sum_{i=1}^{n} x_i^2$ distributed?

$$p(x) = \int d\mathbf{x}\, p(x, \mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x}) \prod_{i=1}^{n} p(x_i) \qquad \text{insert!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dx_1 \ldots \int dx_n\, \delta\Big(x - \textstyle\sum_{i=1}^{n} x_i^2\Big)\, \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} \qquad \text{polar coordinates!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dr \int d\Omega\, r^{n-1}\, \delta\Big(x - r^2\Big)\, \mathrm{e}^{-\frac{1}{2}r^2} \qquad \text{angle integration trivial!}$$

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ with $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{O}, \mathbf{I})$. How is $x = \sum_{i=1}^{n} x_i^2$ distributed?

$$p(x) = \int d\mathbf{x}\, p(x, \mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x}) \prod_{i=1}^{n} p(x_i) \qquad \text{insert!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dx_1 \ldots \int dx_n\, \delta\Big(x - \textstyle\sum_{i=1}^{n} x_i^2\Big)\, \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} \qquad \text{polar coordinates!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dr \int d\Omega\, r^{n-1}\, \delta\Big(x - r^2\Big)\, \mathrm{e}^{-\frac{1}{2}r^2} \qquad \text{angle integration trivial!}$$

$$= (2\pi)^{-\frac{n}{2}} \frac{\pi^{\frac{n}{2}}}{\frac{1}{2}\Gamma(\frac{n}{2})} \int dr\, r^{n-1}\, \delta\Big(x - r^2\Big)\, \mathrm{e}^{-\frac{1}{2}r^2} \qquad \text{substitution: } r = \sqrt{y},\, dr = \frac{dy}{2\sqrt{y}}$$

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ with $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{O}, \mathbf{I})$. How is $x = \sum_{i=1}^{n} x_i^2$ distributed?

$$p(x) = \int d\mathbf{x}\, p(x, \mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x}) \prod_{i=1}^{n} p(x_i) \qquad \text{insert!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dx_1 \ldots \int dx_n\, \delta\Big(x - \textstyle\sum_{i=1}^{n} x_i^2\Big)\, \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} \qquad \text{polar coordinates!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dr \int d\Omega\, r^{n-1}\, \delta\Big(x - r^2\Big)\, \mathrm{e}^{-\frac{1}{2}r^2} \qquad \text{angle integration trivial!}$$

$$= (2\pi)^{-\frac{n}{2}} \frac{\pi^{\frac{n}{2}}}{\frac{1}{2}\Gamma(\frac{n}{2})} \int dr\, r^{n-1}\, \delta\Big(x - r^2\Big)\, \mathrm{e}^{-\frac{1}{2}r^2} \qquad \text{substitution: } r = \sqrt{y},\ dr = \frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \int dy\, y^{\frac{n}{2}-1}\, \delta\Big(x - y\Big)\, \mathrm{e}^{-\frac{1}{2}y} \qquad \text{where: } f(x) = \int dy\, \delta(x - y)\, f(y)$$

Fraunhofer FKIE

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ with $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mathbf{O}, \mathbf{I})$. How is $x = \sum_{i=1}^{n} x_i^2$ distributed?

$$p(x) = \int d\mathbf{x}\, p(x, \mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, p(x|\mathbf{x}) \prod_{i=1}^{n} p(x_i) \qquad \text{insert!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dx_1 \ldots \int dx_n\, \delta\Big(x - \textstyle\sum_{i=1}^{n} x_i^2\Big)\, \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} \qquad \text{polar coordinates!}$$

$$= (2\pi)^{-\frac{n}{2}} \int dr \int d\Omega\, r^{n-1}\, \delta\Big(x - r^2\Big)\, \mathrm{e}^{-\frac{1}{2}r^2} \qquad \text{angle integration trivial!}$$

$$= (2\pi)^{-\frac{n}{2}} \frac{\pi^{\frac{n}{2}}}{\frac{1}{2}\Gamma(\frac{n}{2})} \int dr\, r^{n-1}\, \delta\Big(x - r^2\Big)\, \mathrm{e}^{-\frac{1}{2}r^2} \qquad \text{substitution: } r = \sqrt{y},\ dr = \frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} \int dy\, y^{\frac{n}{2}-1}\, \delta\Big(x - y\Big)\, \mathrm{e}^{-\frac{1}{2}y} \qquad \text{where: } f(x) = \int dy\, \delta(x - y)\, f(y)$$

Answer:

$$p(x) = \frac{x^{\frac{n}{2}-1}\, \mathrm{e}^{-\frac{1}{2}x}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}$$

Excursus: χ^2-distributed random variables

nD Gaussian RV $\mathbf{x}$ with expectation $\mathbb{E}[\mathbf{x}] = \bar{\mathbf{x}}$ and covariance $\mathbb{C}[\mathbf{x}] = \mathbf{P}$

What is the pdf of the derived scalar RV $\boxed{q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}})}$?

Consider transformed RV $\mathbf{u} = \mathbf{P}^{-1/2}(\mathbf{x} - \bar{\mathbf{x}})$! We have: $p(\mathbf{u}) = \mathcal{N}\big(\mathbf{u};\, \mathbf{o},\, \mathbf{I}\big)$

obviously: $q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) = \mathbf{u}^\top \mathbf{u} = \sum_{i=1}^{n} u_i^2$ with: $u_i \sim N(0, 1)$

the pdf of q is: $\boxed{p(q) = \frac{q^{\frac{n-1}{2}} \mathrm{e}^{-\frac{q}{2}}}{2^{\frac{n}{2}} \Gamma(n/2)}, \quad q \sim \chi_n^2}$

'χ^2 with n degrees of freedom'
$\Gamma(\frac{1}{2}) = \sqrt{\pi},\ \Gamma(1) = 1$
$\Gamma(m+1) = m\Gamma(m)$
especially: $\mathbb{E}[q] = n,\ \mathbb{V}[q] = 2n$

Fraunhofer FKIE

Excursus: χ^2-distributed random variables

nD Gaussian RV $\mathbf{x}$ with expectation $\mathbb{E}[\mathbf{x}] = \bar{\mathbf{x}}$ and covariance $\mathbb{C}[\mathbf{x}] = \mathbf{P}$

What is the pdf of the derived scalar RV $\boxed{q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}})}$?

Consider transformed RV $\mathbf{u} = \mathbf{P}^{-1/2}(\mathbf{x} - \bar{\mathbf{x}})$! We have: $p(\mathbf{u}) = \mathcal{N}\big(\mathbf{u};\, \mathbf{o},\, \mathbf{I}\big)$

obviously: $q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) = \mathbf{u}^\top \mathbf{u} = \sum_{i=1}^{n} u_i^2$ with: $u_i \sim N(0, 1)$

the pdf of q is: $\boxed{p(q) = \frac{q^{\frac{n-1}{2}} \mathrm{e}^{-\frac{q}{2}}}{2^{\frac{n}{2}} \Gamma(n/2)}, \quad q \sim \chi^2_n}$

'χ^2 with n degrees of freedom'
$\Gamma(\frac{1}{2}) = \sqrt{\pi},\ \Gamma(1) = 1$
$\Gamma(m+1) = m\Gamma(m)$
especially: $\mathbb{E}[q] = n,\ \mathbb{V}[q] = 2n$

for *independent* RVs $q_1 \sim \chi^2_{n_1}$, $q_2 \sim \chi^2_{n_2}$: $\quad q_1 + q_2 \sim \chi^2_{n_1+n_2}$

Excursus: χ^2-distributed random variables

nD Gaussian RV $\mathbf{x}$ with expectation $\mathbb{E}[\mathbf{x}] = \bar{\mathbf{x}}$ and covariance $\mathbb{C}[\mathbf{x}] = \mathbf{P}$

What is the pdf of the derived scalar RV $\boxed{q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}})}$?

Consider transformed RV $\mathbf{u} = \mathbf{P}^{-1/2}(\mathbf{x} - \bar{\mathbf{x}})$! We have: $p(\mathbf{u}) = \mathcal{N}\big(\mathbf{u};\, \mathbf{o},\, \mathbf{I}\big)$

obviously: $q = (\mathbf{x} - \bar{\mathbf{x}})^\top \mathbf{P}^{-1} (\mathbf{x} - \bar{\mathbf{x}}) = \mathbf{u}^\top \mathbf{u} = \sum_{i=1}^{n} u_i^2$ with: $u_i \sim N(0, 1)$

the pdf of q is: $\boxed{p(q) = \frac{q^{\frac{n-1}{2}} \mathrm{e}^{-\frac{q}{2}}}{2^{\frac{n}{2}} \Gamma(n/2)}, \quad q \sim \chi_n^2}$

'χ^2 with n degrees of freedom'
$\Gamma(\frac{1}{2}) = \sqrt{\pi},\ \Gamma(1) = 1$
$\Gamma(m+1) = m\Gamma(m)$
especially: $\mathbb{E}[q] = n,\ \mathbb{V}[q] = 2n$

for *independent* RVs $q_1 \sim \chi^2_{n_1}$, $q_2 \sim \chi^2_{n_2}$: $\quad q_1 + q_2 \sim \chi^2_{n_1+n_2}$

$Q(\lambda|n) = P(q > \lambda) = \int_\lambda^\infty dq\, p(q) \quad \rightarrow \quad$ correlation probability: $\quad P_c = 1 - Q(\lambda|n)$

λ can be taken from a χ^2 table!

A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

$$p\big(\mathbf{z}_k|\,\mathcal{Z}^{k-1}\big) = \int d\mathbf{x}_k\; p\big(\mathbf{z}_k|\,\mathbf{x}_k\big)\; p\big(\mathbf{x}_k|\,\mathcal{Z}^{k-1}\big)$$

$$= \int d\mathbf{x}_k\; \underbrace{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)}_{\text{likelihood: sensor model}}\; \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}_{\text{prediction at time } t_k}$$

$$= \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_{k|k-1},\, \mathbf{S}_{k|k-1}\big) \qquad \text{(product formula)}$$

innovation:	$\boldsymbol{\nu}_{k\|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k\|k-1},$
innovation covariance:	$\mathbf{S}_{k\|k-1} = \mathbf{H}_k\mathbf{P}_{k\|k-1}\mathbf{H}_k^\top + \mathbf{R}_k$
expectation gate:	$\boldsymbol{\nu}_{k\|k-1}^\top \mathbf{S}_{k\|k-1}^{-1} \boldsymbol{\nu}_{k\|k-1} \le \lambda(P_c)$
MAHALANOBIS:	ellipsoid containing $\mathbf{z}_k$ with probability P_c
	$\lambda(P_c)$: gating parameter ($\rightarrow \chi^2$-table)

$n\backslash Q$	0.99	0.975	0.95	0.90	0.75	0.5	0.25	0.10	0.05	0.025	0.01	5E-3	1E-3
1	2E-4	.001	.003	.016	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88	10.8
2	.020	.051	.103	.211	.575	.139	.277	4.61	5.99	7.38	9.21	10.6	13.8
3	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8	16.3
4	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9	18.5
5	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7	20.5
6	.872	1.24	1.64	2.20	3.35	5.35	7.84	10.6	12.6	14.4	16.8	18.5	22.5
7	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.1	18.5	20.3	24.3
8	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0	26.1
9	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	17.0	19.0	21.7	23.6	27.9
10	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2	29.6
11	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	22.0	24.7	26.8	31.3
12	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3	32.9
13	4.11	5.01	5.90	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8	34.5
14	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3	36.1
15	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8	37.7
16	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3	39.3
17	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7	40.8
18	7.01	8.23	9.40	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2	42.3
19	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6	43.8
20	8.26	9.60	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0	45.3
25	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9	52.6
30	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7	59.7
40	22.2	24.4	26.5	29.1	33.7	39.3	45.6	51.8	55.8	59.3	63.7	66.8	73.4
50	29.7	32.4	34.8	37.7	43.0	49.3	56.3	63.2	67.5	71.4	76.2	79.5	86.7
60	37.5	40.5	43.2	46.5	52.3	59.3	67.0	74.4	79.1	83.3	88.4	92.0	99.6
70	45.4	48.8	51.7	55.3	61.7	69.3	77.6	85.5	90.5	95.0	100	104	112
80	53.5	57.2	60.4	64.2	71.1	79.3	88.1	96.6	102	107	112	116	125
90	61.8	65.6	69.1	73.3	80.6	89.3	98.6	108	113	118	124	128	137
100	70.1	74.2	77.9	82.4	90.1	99.3	109	118	124	130	136	140	149
$\mathcal{G}()$	-2.33	-1.96	-1.64	-1.28	-.675	0	.675	1.28	1.64	1.96	2.33	2.58	3.09