Recapitulation: How to deal with pdfs?

Example: Consider a RV with pdf $p(\mathbf{x})$! How to calculate the pdf $p(\mathbf{y})$ of a RV $\mathbf{y} = \mathbf{t}[\mathbf{x}]$ resulting from $\mathbf{x}$ by a transformation $\mathbf{t} : \mathbf{x} \mapsto \mathbf{t}[\mathbf{x}]$?

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$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x}, \mathbf{y}) \qquad \text{marginalize: bring } \mathbf{x} \text{ into the play!}$$

$$= \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) \qquad \text{notion of a conditional pdf}$$

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$$
\begin{aligned}
p(\mathbf{y}) &= \int d\mathbf{x}\, p(\mathbf{x}, \mathbf{y}) && \text{marginalize: bring } \mathbf{x} \text{ into the play!} \\
&= \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) && \text{notion of a conditional pdf} \\
&= \int d\mathbf{x}\, \delta(\mathbf{y} - \mathbf{t}[\mathbf{x}])\, p(\mathbf{x}) && \text{certain knowledge, if } \mathbf{x} \text{ is known!}
\end{aligned}
$$

For Dirac-distributions: “integration formula”: $\int dx\, \delta(x-y)\, f(x) = f(y)$.

Assumption: invertable transformation $\mathbf{t} : \mathbf{x} \mapsto \mathbf{z} = \mathbf{t}[\mathbf{x}]$! Substitute in the integral: $\mathbf{x} = \mathbf{t}^{-1}[\mathbf{z}]$.

Remember: substitution rule for volume integrals

$$\varphi : y \mapsto \varphi[y] = x \qquad \int_a^b dx\, f(x) = \int_{\varphi[a]}^{\varphi[b]} dy\, \frac{d\varphi[y]}{dy}\, f(\varphi[y])$$

$$\boldsymbol{\varphi} : \mathbf{y} \mapsto \boldsymbol{\varphi}[\mathbf{y}] = \mathbf{x} \qquad \int_X d\mathbf{x}\, \mathbf{f}(\mathbf{x}) = \int_{\boldsymbol{\varphi}[X]} d\mathbf{y} \left| \frac{\partial \boldsymbol{\varphi}[\mathbf{y}]}{\partial \mathbf{y}} \right| \mathbf{f}(\boldsymbol{\varphi}[\mathbf{y}])$$

Jacobian $=$ matrix of the first derivatives of a vector-variate function

$\boldsymbol{\varphi} : \mathbf{x} \mapsto \boldsymbol{\varphi}[\mathbf{x}] = (\varphi_1[\mathbf{x}], \dots, \varphi_m[\mathbf{x}])^\top$**,** $\mathbf{x} = (x_1, \dots, x_n)^\top$**:**

$$\boldsymbol{\Phi} = \begin{pmatrix} \frac{\partial \varphi_1[\mathbf{x}]}{\partial x_1} & \dots & \frac{\partial \varphi_m[\mathbf{x}]}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_1[\mathbf{x}]}{\partial x_n} & \dots & \frac{\partial \varphi_m[\mathbf{x}]}{\partial x_n} \end{pmatrix} =: \frac{\partial \boldsymbol{\varphi}[\mathbf{x}]}{\partial \mathbf{x}}$$

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$$= \int d\mathbf{x}\, \delta(\mathbf{y} - \mathbf{t}[\mathbf{x}])\, p(\mathbf{x}) \qquad \text{certain knowledge, if } \mathbf{x} \text{ is known!}$$

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Corresponding Jacobi determinant: $|\mathbf{T}^{-1}(\mathbf{z})|$ mit $\mathbf{T}^{-1}(\mathbf{z})| = \frac{\partial \mathbf{t}^{-1}[\mathbf{z}]}{\partial \mathbf{z}}$.

$$p(\mathbf{y}) = \int d\mathbf{z}\, |\mathbf{T}^{-1}(\mathbf{z})|\, \delta(\mathbf{y} - \mathbf{z})\, p(\mathbf{t}^{-1}[\mathbf{z}])$$
$$= |\mathbf{T}^{-1}(\mathbf{y})|\, p(\mathbf{t}^{-1}[\mathbf{y}]) =: \mathcal{T}[p](\mathbf{y}) \qquad \mathcal{T} \text{ is called “Transfer-Operator”}$$

To be generalized under certain assumptions!

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Let $\mathbf{x}$ be a Gaussian RV with $p(\mathbf{x}) = \mathcal{N}\big(\mathbf{x};\, \mathbb{E}[\mathbf{x}],\, \mathbb{C}[\mathbf{x}]\big)$.
Show for the pdf of the RV $\mathbf{y} = \mathbf{t}[\mathbf{x}]$, resulting from $\mathbf{x}$ by an *affine* transformation $\mathbf{t}[\mathbf{x}] = \mathbf{a} + \mathbf{A}\mathbf{x}$:

Exercise (voluntary)

$$p(\mathbf{y}) = \left|\frac{\partial \mathbf{t}^{-1}[\mathbf{y}]}{\partial \mathbf{y}}\right| p\big(\mathbf{t}^{-1}[\mathbf{y}]\big)$$

$$= \mathcal{N}\big(\mathbf{y};\, \mathbf{a} + \mathbf{A}\mathbb{E}[\mathbf{x}],\, \mathbf{A}\mathbb{E}[\mathbf{x}]\mathbf{A}^\top\big)$$

$\mathbf{a}$, $\mathbf{A}$: vector/matrix of suitable dimension (constant),

$$\mathbf{t}^{-1} : \mathbf{y} \mapsto \mathbf{t}^{-1}[\mathbf{y}] = \mathbf{A}^{-1}(\mathbf{y} - \mathbf{a}), \quad \frac{\partial \mathbf{t}^{-1}[\mathbf{y}]}{\partial \mathbf{y}} = \mathbf{A}^{-1}$$

Remember some rules for dealing with matrices:

$$|\mathbf{A}||\mathbf{B}| = |\mathbf{AB}|,\ |A^{-1}| = |\mathbf{A}|^{-1}$$

$$(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top,\ (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$