

## Exercise 3.1

Zeige, dass gilt:  $\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$ !

Der Erwartungswert der Observablen  $x^2$  heißt auch “2. Moment” der WDF von  $x$

## Exercise 3.1

Zeige, dass gilt:  $\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$ !

mit  $\bar{x} = \mathbb{E}[x]$  gilt: 
$$\mathbb{V}[x] = \int_{-\infty}^{\infty} dx (x - \bar{x})^2 p(x)$$

## Exercise 3.1

Zeige, dass gilt:  $\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$ !

$$\begin{aligned} \text{mit } \bar{x} = \mathbb{E}[x] \text{ gilt: } \quad \mathbb{V}[x] &= \int_{-\infty}^{\infty} dx (x - \bar{x})^2 p(x) \\ &= \int_{-\infty}^{\infty} dx x^2 p(x) - 2\bar{x} \int_{-\infty}^{\infty} dx x p(x) + \bar{x}^2 \int_{-\infty}^{\infty} dx p(x) \end{aligned}$$

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$$\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

mit  $\bar{x} = \mathbb{E}[x]$  gilt: 
$$\mathbb{V}[x] = \int_{-\infty}^{\infty} dx (x - \bar{x})^2 p(x)$$

$$= \underbrace{\int_{-\infty}^{\infty} dx x^2 p(x)}_{=\mathbb{E}[x^2]} - 2\bar{x} \underbrace{\int_{-\infty}^{\infty} dx x p(x)}_{=\bar{x} \text{ Erwartung}} + \bar{x}^2 \underbrace{\int_{-\infty}^{\infty} dx p(x)}_{=1 \text{ Normierung}}$$

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## Exercise 3.1

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$$\begin{aligned} \text{mit } \bar{x} = \mathbb{E}[x] \text{ gilt: } \quad \mathbb{V}[x] &= \int_{-\infty}^{\infty} dx (x - \bar{x})^2 p(x) \\ &= \underbrace{\int_{-\infty}^{\infty} dx x^2 p(x)}_{=\mathbb{E}[x^2]} - 2\bar{x} \underbrace{\int_{-\infty}^{\infty} dx x p(x)}_{=\bar{x} \text{ Erwartung}} + \bar{x}^2 \underbrace{\int_{-\infty}^{\infty} dx p(x)}_{=1 \text{ Normierung}} \\ &= \mathbb{E}[x^2] - 2\bar{x}^2 + \bar{x}^2 = \mathbb{E}[x^2] - \mathbb{E}[x]^2 \quad \diamond \end{aligned}$$

Der Erwartungswert der Observablen  $x^2$  heißt auch “2. Moment“ der WDF von  $x$

Berechne Erwartung und Varianz der Gleichverteilung einer ZV  $x \in \mathbb{R}$  im Intervall  $[a, b]$

### Exercise 3.2

$$p(x) = \mathcal{U}(\underbrace{x}_{\text{ZV}}; \underbrace{a, b}_{\text{Parameter}}) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

Ist die WDF richtig normiert?  $\int_{-\infty}^{\infty} dx \mathcal{U}(x; a, b) = 1$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx x \mathcal{U}(x; a, b) = \frac{b+a}{2}$$

$$\mathbb{V}[x] = \frac{1}{b-a} \int_a^b dx x^2 - \mathbb{E}[x]^2 = \frac{1}{12}(b-a)^2$$

Berechne Erwartung und Varianz der Gleichverteilung

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Berechne Erwartung und Varianz der Gleichverteilung

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$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx x \mathcal{U}(x; a, b) = \frac{1}{b-a} \int_a^b dx x = \frac{\frac{1}{2}x^2 \Big|_a^b}{b-a} = \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{b+a}{2}$$

Berechne Erwartung und Varianz der Gleichverteilung

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$$\mathbb{V}[x] = \frac{1}{b-a} \int_a^b dx x^2 - \mathbb{E}[x]^2 \quad \text{Exercise 3.1!}$$

Berechne Erwartung und Varianz der Gleichverteilung

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$$\mathbb{V}[x] = \frac{1}{b-a} \int_a^b dx x^2 - \mathbb{E}[x]^2 = \frac{\frac{1}{3}b^3 - a^3}{b-a} - \frac{1}{4}(b+a)^2$$

## Berechne Erwartung und Varianz der Gleichverteilung

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$$\begin{aligned} \mathbb{V}[x] &= \frac{1}{b-a} \int_a^b dx x^2 - \mathbb{E}[x]^2 = \frac{1}{3} \frac{b^3 - a^3}{b-a} - \frac{1}{4}(b+a)^2 \\ &= \frac{4b^3 - 4a^3 - 3b^3 - 3ab^2 + 3a^2b + 3a^3}{12(b-a)} \end{aligned}$$

## Berechne Erwartung und Varianz der Gleichverteilung

### Exercise 3.2

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## Berechne Erwartung und Varianz der Gleichverteilung

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$$\begin{aligned} \mathbb{V}[x] &= \frac{1}{b-a} \int_a^b dx x^2 - \mathbb{E}[x]^2 = \frac{1}{3} \frac{b^3 - a^3}{b-a} - \frac{1}{4}(b+a)^2 \\ &= \frac{4b^3 - 4a^3 - 3b^3 - 3ab^2 + 3a^2b + 3a^3}{12(b-a)} = \frac{(b-a)^3}{12(b-a)} = \frac{1}{12}(b-a)^2 \end{aligned}$$

## Exercise 2.3

Show for the Gauss density  $p(x) = \mathcal{N}(x; \mu, \sigma)$ :

$$\mathbb{E}[x] = \mu, \quad \mathbb{V}[x] = \sigma^2$$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx \, x \mathcal{N}(x; \mu, \sigma) = \mu$$

$$\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2$$

Solution via substitution and partielle integration!

$$\text{Use } \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}!$$

$$\mathbb{E}[x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x - \mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma dy$$



$$\begin{aligned}\mathbb{E}[x] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy (\sigma y + \mu) e^{-\frac{1}{2}y^2}\end{aligned}$$

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\mathbb{E}[x] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy (\sigma y + \mu) e^{-\frac{1}{2}y^2} \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy y e^{-\frac{1}{2}y^2} - \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy y e^{-\frac{1}{2}y^2} + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-\frac{1}{2}y^2} = \mu
\end{aligned}$$

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\end{aligned}$$

$$\mathbb{E}[x^2] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy (\sigma y + \mu)^2 e^{-\frac{1}{2}y^2}$$

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&= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy y^2 e^{-\frac{1}{2}y^2} + \mu^2, \quad \mathbb{V}[x] = \mathbb{E}[x^2] - (\mathbb{E}[x])^2
\end{aligned}$$

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\mathbb{E}[x] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma y \\
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&= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy y^2 e^{-\frac{1}{2}y^2} + \mu^2, \quad (e^{-\frac{1}{2}y^2})' = -y e^{-\frac{1}{2}y^2}
\end{aligned}$$

partial integration:  $\int_a^b dx g(x) f'(x) = g(x) f(x) \Big|_a^b - \int_a^b dx g'(x) f(x)$

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\mathbb{E}[x] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx x e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma y \\
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\end{aligned}$$

partial integration:  $\int_a^b dx g(x) f'(x) = g(x) f(x) \Big|_a^b - \int_a^b dx g'(x) f(x)$

$$\mathbb{V}[x^2] = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy y \left( y e^{-\frac{1}{2}y^2} \right) = \frac{-\sigma^2}{\sqrt{2\pi}} \underbrace{\left( y e^{-\frac{1}{2}y^2} \right) \Big|_{-\infty}^{\infty}}_{\rightarrow 0 \text{ symmetry!}} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-\frac{1}{2}y^2} = \sigma^2$$