Advanced Sensor Data Fusion in Distributed Systems

SS 2018
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About me

Research interest:

- State estimation and target tracking in multi sensor networks
- In particular
  - Out-of-Sequence Processing and
  - Track-to-Track Fusion
  - Tensor Decomposition Tracking
- Diploma of Mathematics (2007)
- Ph.D. of Computer Science (2012)
- University of Bonn (2007 – 2009)
- Fraunhofer FKIE (since 2009)
  - Former Head of Research Group “Distributed Systems”
  - Deputy of Head of Department “Sensor Data and Information Fusion”
Our institute

FRAUNHOFER FKIE
Department: Sensor Data and Information Fusion

FG Integrated Systems
D. Bender

FG Resource Management
A. Charlish

Sensor Data and Information Fusion (SDF)
AL: W. Koch

FG Distributed Systems
R. Zemmari, M. Ulmke

FG Array Processing
M. Oispuu
A motivation for

SENSOR DATA FUSION IN DISTRIBUTED SYSTEMS
All Creatures do Sensor Data Fusion (SDF)

- Ambiguous sensing information
- Sensing error
- Complementary sensors
  - visual
  - acoustic
  - smelling
  - etc.
- Environment is a dense situation
- Target behavior can be predicted

Sensor Data Fusion is a cognitive tool for situation awareness.
SDF for Technical Sensors

Dense information about an object of interest. -> Tracks

Sensor information implies challenges:

- Measurement error
- False returns (clutter)
- Dynamic scenario
- Data association
- Resolution conflicts
- Doppler blindness
- Maneuvers
Distributed Sensor

\[ x_{k\mid k} = E \left[ x_{k\mid k}, x_{k\mid k}^1, \ldots, x_{k\mid k}^5 \right] =? \]

\[ P_{k\mid k} = cov \left[ x_{k\mid k}, x_{k\mid k}^1, \ldots, x_{k\mid k}^5 \right] =? \]
Centralized vs. Decentralized vs. Distributed

Centralized:
- Distinguished Fusion Center (FC)
- Send all measurements to FC
- No cross covariances of tracks
- Doesn’t scale with False Alarms (FA)

Decentralized:
- Every node is a FC
- Send data only to neighbours
- Estimate network topology
- Account for “information incest”
- Only approximate solutions exist

Distributed:
- Distinguished FC
- Preprocess measurements on nodes
- Send estimates to FC
- Account for X-correlations
- Scales very well in number of sensors and in FA.
Tracking in Distributed Systems

Distributed Systems have the following advantages and drawbacks.

Advantages:
- No single point of failure
- Cheap sensor technology
- Complementary sensors
- Spatial distribution
- Distributed computation

Challenges:
- Communication of data necessary
- Sensor synchronization
- Sensor registration
- Correlations
Hardware, Data Flow, and Software

GAMMA2

Echse2

RawDataDisk (IQ data)

Signal Processing (Laptops)

Disk (plots)

Tracking (Laptop)

Base Station Data Base

UserDisplay

Advanced SDF in Distributed Systems #1
SS 2018
How to describe

INSECURE KNOWLEDGE
Probability Density Functions (pdf)

- Pdfs represent imprecise knowledge.
- Pdf $p()$ is a real valued function and $p \geq 1$.
- $p$ is normalized
  \[ \int dx p(x) = 1 \]
- Imprecise knowledge about the state of an object at time $k$ given all data up to time $k$:
  \[ p(x_k | Z^k) \]
- Probability that the state is within a region $\mathbf{U}$:
  \[ \int_{\mathbf{U}} dx p(x_k | Z^k) \]
pdf of prediction at time $t_k$
pdf after update with single plot
pdf after update with two plots
pdf after “merging” ambiguous modes
STOCHASTIC MOTION
A stochastic process is a set of random variables, which represent the state of a non-deterministic state:

**Continuous**

\[ \{x_t\}_{t \geq 0} \]

\( t = \text{time} \)

**Discrete**

\[ \{x_k\}_{k \in \mathbb{N}} \]

\( k = \text{time step} \)
Example

Discrete, $\Delta t = 1/8$
Stochastic Motion: Diffusion in 1D

For a state $x_k$ in $\mathbb{R}$, consider the following transition kernel:

$$p(x_k|x_{k-1}) = \begin{cases} \frac{1}{2}, & \text{if } |x_k - x_{k-1}| = h \\ 0, & \text{else.} \end{cases}$$

This implies for the pdf of the succeeding state that

$$p(x_{k+1}) = \sum_{x_k \in h\mathbb{Z}} p(x_{k+1}|x_k)p(x_k) = \frac{1}{2} p(x_k + h) + \frac{1}{2} p(x_k - h).$$

As a consequence, one has

$$\frac{p(x_{k+1}) - p(x_k)}{\tau} = \frac{h^2}{2\tau} \frac{p(x_k + h) - 2p(x_k) + p(x_k - h)}{h^2} \quad h \to 0 \text{ and } \tau \to 0$$

$$\partial_t p(x_k) = D \partial^2_x p(x_k)$$

$D = \frac{h^2}{2\tau}$

$\tau = \text{time step}$
BAYESIAN ESTIMATION
Basic Idea of Sensor Data Fusion

Prior knowledge: target dynamics model, sensor model, context information.

- **prediction:**  
  \[ p(x_{k-1}\mid \mathcal{Z}^{k-1}) \xrightarrow{\text{dynamics model}} \rightarrow \text{road maps} \xrightarrow{} p(x_k\mid \mathcal{Z}^{k-1}) \]

- **filtering:**  
  \[ p(x_k\mid \mathcal{Z}^{k-1}) \xrightarrow{\text{sensor data } \mathcal{Z}_k} \rightarrow \text{sensor model} \xrightarrow{} p(x_k\mid \mathcal{Z}^{k}) \]

- **retrodiction:**  
  \[ p(x_{l-1}\mid \mathcal{Z}^{k}) \xleftarrow{\text{filtering output}} \rightarrow \text{dynamics model} \xrightarrow{} p(x_l\mid \mathcal{Z}^{k}) \]
Prediction – Filtering - Cycle

Retrodiction (optional) → Predicting → Filtering → Retrodiction (optional)
Difficult Operational Conditions

- Target detection
  - Small objects \( P_D < 1 \)
  - Fading
  - Minimum detectable radial velocity
- Measurements
  - False returns (clutter, birds, clouds, waves)
  - Measurement error
  - Data to target assignment
- Sensor resolution
  - Group measurements
  - Resolution probability model
- Target behavior
  - High maneuvering capability
  - Distinct maneuvering phases
  - Dynamic object parameters unknown
Conclusion: Approach of Bayesian Target Tracking

- **Basis:** Measurements of a target of interest.
- **Objective:** Gather as much information as possible by analyzing measurements.
- **Challenge:** Imperfect sensor information and target state evolution in time.
- **Approach:** Interpret target state as Random Variable (RV) and infer pdf about what is known.
- **Solution:** Derive iterative formula for calculating pdfs.
  - + mechanism for state initialization
  - + exploit background information
- **Parameters of interest:** estimate and (co-) variance.
How to deal with pdfs?

- **conditional pdf**
  \[ p(x|y) = \frac{p(x,y)}{p(y)} \]

  Impact of knowledge about a RV y on x

- **marginalization**
  \[ p(x) = \int dy \ p(x,y) = \int dy \ p(x|y) \ p(y) \]

  Introduce y to the equation

- **Bayes Theorem**
  \[ p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx \ p(y|x)p(x)} \]

  Posterior = c x Likelihood function x Prior

- **Certain knowledge**
  \[ p(x) = \delta(x - y) \]

  Certain knowledge that x = y
Examples of a Target State

- Position and velocity
  \[ \mathbf{x}_k = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} \]

- Position, velocity and acceleration
  \[ \mathbf{x}_k = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{pmatrix} \]

- Kinematics and classification
  \[ \mathbf{x}_k = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \\ \cdots \\ c \end{pmatrix} \]

- Single parameter estimation
  \[ \mathbf{x}_k = (x) \]
Parameters of Interest: Estimate and Covariance

The expectation of a pdf is its centroid:

$$E[x] = \int_{\mathbb{R}^n} dx \ p(x) \cdot x$$

The covariance is a measure of spreading around the expectation value:

$$\text{cov}[x] = E[(x - E[x])^2]$$

$$\bar{x} = E[x]$$

$$\text{cov}[x] = E[(x - \bar{x}) \cdot (x - \bar{x})^\top]$$
Important Family of PDFs

- wanted: Probability mass concentrated around a center \( \mu \).
- Parameter for width of the pdf: \( \sigma \)

\[
\mathcal{N}(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right)
\]

- Calculations show

\[
\mathbb{E}[x] = \int_{-\infty}^{\infty} dx \ x \mathcal{N}(x; \mu, \sigma) = \mu
\]

\[
\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2
\]
Delta Peak: a Limiting Case of a Gaussian

- Secure knowledge: delta peaks.
  \[ \delta(x; \mu) = \begin{cases} \infty & x = \mu \\ 0 & x \neq \mu \end{cases} \]

- Can be interpreted as a limiting case:
  \[ \delta(x; \mu) \xrightarrow{\sigma \to 0} \lim_{\sigma \to 0} \mathcal{N}(x; \mu, \sigma) \]
Generalization to Multivariate (MV) RVs

- Accumulate parameters of interest into a vector state:
  \[ \mathbb{R}^n \ni \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \]

- Probability can be obtained via volume integral:
  \[ P\{\mathbf{x} \in V\} = \int_V dx_1 \ldots dx_n \, p(x_1, \ldots, x_n) \]

For example:

\[ x_1 = x \]
\[ x_2 = y \]
\[ x_3 = \dot{x} \]
\[ x_4 = \dot{y} \]
Expectation and Covariance of a MV Gaussian

- A MV Gaussian is defined as

\[
p(x) = \mathcal{N}(x; \bar{x}, P) = \frac{1}{\sqrt{\det(2\pi P)}} e^{-\frac{1}{2}(x-\bar{x})^T P^{-1}(x-\bar{x})}
\]

- where
  - \( \mathbb{E}[x] = \bar{x} \in \mathbb{R}^n \)
  - \( \text{cov}[x] = P \in \mathbb{R}^{n \times n} \)
Correlation of Random Variables

Let $x$ and $y$ be real valued random variables with a normal distributed joint density:

$$p(x, y) = \mathcal{N}\left(\left(x, y\right); \left(\bar{x}, \bar{y}\right), \begin{pmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{pmatrix}\right)$$

The covariance matrix consists of

$$P_{xx} = \text{cov} \left[ x \right] = \text{cov} \left[ x, x \right] = E \left[ (x - E \left[ x \right])^2 \right]$$

$$P_{xy} = \text{cov} \left[ x, y \right] = E \left[ (x - E \left[ x \right])(y - E \left[ y \right]) \right]$$

Assume $x$ and $y$ are zero-mean. That is

$$E \left[ x \right] = E \left[ y \right] = 0.$$

Then:

$$P_{xx} = E \left[ x^2 \right]$$

$$P_{xy} = E \left[ x \cdot y \right]$$
Uncorrelated and Independent Random Variables

If the joint density has the following form:

\[ p(x, y) = \mathcal{N} \left( \left( \begin{array}{c} x \\ y \end{array} \right); \left( \begin{array}{c} \bar{x} \\ \bar{y} \end{array} \right), \left( \begin{array}{cc} P_{xx} & 0 \\ 0 & P_{yy} \end{array} \right) \right) \]

Then \( x \) and \( y \) are called „uncorrelated”.

If the joint density has the following form:

\[ p(x, y) = p(x) \cdot p(y) \]

Then \( x \) and \( y \) are called „independent”.

Exercise

If \( x \) and \( y \) are normal distributed and uncorrelated, then they are independent!
Structure of a Covariance Matrix

Let $\mathbf{x}$ be given as $\mathbb{R}^n \ni \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

and $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \bar{x}, \mathbf{P})$

Then:

$$\mathbf{P} = (\text{cov} [x_i, x_j])_{i,j=1,\ldots,n}$$

in particular:

$$\mathbf{P} = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \ldots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \ldots & P_{2n} \\ P_{31} & P_{32} & P_{33} & \ldots & P_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & P_{n3} & \ldots & P_{nn} \end{pmatrix}$$

Note that:

- Diagonal elements are the variance of parameter $x_i$
- Off-diagonal elements are the correlation of $x_i$ and $x_j$
- $\mathbf{P}$ has the following properties:
  - symmetric $P_{ij} = P_{ji}$
  - positive definite
    $$\forall \mathbf{x} \neq 0 : \mathbf{x}^\top \mathbf{P} \mathbf{x} > 0$$
  - has a positive determinant:
    $$|\mathbf{P}| := \det(\mathbf{P}) > 0$$
Example 1

\[ P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
Example 2

\[
P = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}
\]
Example 3

\[ P = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix} \]
Example 4

\[ P = \begin{pmatrix} 0.1 & 0.3 \\ 0.3 & 1 \end{pmatrix} \]
Statistical Models for
SENSORS
Linear Gaussian Case

- The measurement $z$ at time $k$ is modeled as a RV.
- Relationship to state is described by the "measurement equation":
  \[ z_k = H_k x_k + v_k \]

where:
- $H$ is the *measurement function*
- $v$ is the measurement noise.
Ideal Sensor for Tracking

An ideal sensor is given if

- $H$ is linear.
- $v$ is zero-mean and Gaussian distributed, that is
  - $E[v_k] = 0$
  - $p(v_k) = \mathcal{N}(v_k; 0, R_k)$
  - in short: $v_k \sim \mathcal{N}(v_k; 0, R_k)$
- \[ \iff z_k \sim \mathcal{N}(z_k; H_k x_k, R_k) = p(z_k|x_k) \]

Why is this sensor ideal?
If transition model is also linear and Gaussian, an optimal closed solution can be given -> Kalman filter
Example 1

Assume the state is given by

\[ x_k = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} \]

then, the measurement equation for a position sensor (GPS e.g.) is

\[ z_k = H_k(x_k) + v_k \]

\[ = \begin{pmatrix} x \\ y \end{pmatrix} + \text{noise} \]

therefore

\[ H_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]
Non-linear Sensor Model

What can we do, if we have a non-linear measurement function:

\[ z_k = h(x_k) + v_k \]

- What to do then?
  - e.g. “Be wise, linearize!”

\[ \tilde{H}_k := \left. \frac{dh(x)}{dx} \right|_{x=x_k} \]

- More details in section “Non-linear filtering”.

![Diagram showing non-linear function and linearized approximation]
Example 2

Assume the state is given by

\[
\mathbf{x}_k = \begin{pmatrix}
x \\
y \\
z \\
x \\
y \\
z
\end{pmatrix} \in \mathbb{R}^6
\]

then, the measurement equation of a air surveillance radar is

\[
\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k
\]

\[
= \begin{pmatrix}
\text{range } r \\
\text{bearing } \alpha \\
\text{elevation } \epsilon
\end{pmatrix} + \text{noise}
\]

\[
= \begin{pmatrix}
\sqrt{x^2 + y^2 + z^2} \\
\arctan\left(\frac{y}{x}\right) \\
\arctan\left(\frac{z}{x}\right)
\end{pmatrix} + \text{noise}
\]
Non-Gaussian Noise

What can be done, if the noise is not Gaussian?

- Approximate it by a Gaussian
- Approximate it by a Gaussian Mixture
  \[ z_k \sim \sum_{i=1}^{I} \mathcal{N}(z_k; \bar{z}_k, R_k^i) \]
- Use a particle representation
  \[ z_k \sim \sum_{i=1}^{N} w_i \delta(z_i - z_k) \]
- See more in the lecture on “Particle Filters”
Statistical models for DYNAMICS
What is a Dynamics Model?

A dynamics model is a transition function

- from the state $x_{k-1}$ at time $k-1$
- to the state $x_k$ at time $k$.

Dynamics (incl. maneuvers) are often insecure knowledge.

Use pdfs to model kinematic evolution in time!
Linear Gaussian Case

The system equation is the mathematical description of the system evolution in time:

\[ x_k = F_{k|k-1} x_{k-1} + w_k \]

where:
- \( F \) is the evolution matrix
- \( w \) is the evolution noise.

Ideal (Kalman filter) case is if
- \( F \) is a linear function (matrix)
- \( w_k \sim \mathcal{N}(w_k; 0, Q_{k|k-1}) \)
- i.e.

\[ p(x_k|x_{k-1}) = \mathcal{N}(x_k; F_{k|k-1} x_{k-1}, Q_{k|k-1}) \]
Non-linear Dynamics

If the evolution is described by a non-linear function

\[ x_k = f_{k|k-1}(x_{k-1}) + w_k \]

Analogously to the non-linear sensor model, an approximation is given by the linearization;

\[ \tilde{F}_{k|k-1} := \left. \frac{df_{k|k-1}(x)}{dx} \right|_{x=x_{k-1}} \]
Example 3

Assume the state $\mathbf{x}$ at time $k-1$ is given by

$$
\mathbf{x}_{k-1} = \begin{pmatrix}
  x_{k-1} \\
  y_{k-1} \\
  \dot{x}_{k-1} \\
  \dot{y}_{k-1} \\
  \ddot{x}_{k-1} \\
  \ddot{y}_{k-1}
\end{pmatrix}
$$

and let the time difference between stage $k$ and $k-1$ be $\Delta t$

$$
x_k = x_{k-1} + \Delta t \dot{x}_{k-1} + \frac{1}{2} \Delta t \ddot{x}_{k-1}
$$

$$
y_k = y_{k-1} + \Delta t \dot{y}_{k-1} + \frac{1}{2} \Delta t \ddot{y}_{k-1}
$$

$$
\dot{x}_k = \dot{x}_{k-1} + \Delta t \ddot{x}_{k-1}
$$

$$
\dot{y}_k = \dot{y}_{k-1} + \Delta t \ddot{y}_{k-1}
$$

$$
\ddot{x}_k = \ddot{x}_{k-1}
$$

$$
\ddot{y}_{k-1} = \ddot{y}_{k-1}
$$

$$
F_{k|k-1} = \begin{pmatrix}
  1 & 0 & \Delta t & 0 & \frac{1}{2} \Delta t & 0 \\
  0 & 1 & 0 & \Delta t & 0 & \frac{1}{2} \Delta t \\
  0 & 0 & 1 & 0 & \Delta t & 0 \\
  0 & 0 & 0 & 1 & 0 & \Delta t \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$
Example 4

Assume a state is given in range-bearing-elevation-space

\[ \mathbf{x}_{k-1} = \begin{pmatrix} r \\ \alpha \\ \epsilon \\ \dot{r} \\ \dot{\alpha} \\ \dot{\epsilon} \end{pmatrix} \]

Then, the state transition is a non-linear function.

\[ f_{k|k-1} (\mathbf{x}_{k-1}) = \]

Transform into Cartesian coordinates

“predict”

Transform into polar coordinates
The evolution of a stochastic system can be described as a first-order partial differential equation (pde).

\[ \dot{x}(t) = A(t)x(t) + D(t)v(t) \]

Where

- **A** is the system matrix
- **D** is the noise gain matrix
- **v** is the continuous process noise

The solution of this pde is given by

\[ x(t) = F(t_0, t)x(t_0) + \int_{t_0}^{t} d\tau F(\tau, t)D(\tau)v(\tau) \]

\[ F(t_0, t) = \exp(\Delta t A) \]
Example 3b

The state-space representation of example 3 has the following form:

\[ \dot{x}(t) = A(t)x(t) + D(t)v(t) \]

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\ddot{x} \\
\ddot{y}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
\dot{x} \\
\dot{y}
\end{pmatrix}
+ 
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix} \cdot q_v
\]
Discretized State-Space Noise

The discretized evolution is given by.
\[ x_k = F_{k|k-1} x_{k-1} + w_k \]

From the pde solution, we know that:
\[ w_k = \int_{t_0}^{t} d\tau \ F(\tau, t) D(\tau) v(\tau) \]

Therefore:
\[ w_k = q_v \int_{t_0}^{t} d\tau \ \Gamma(\Delta t) \]

where \[ \Gamma(\Delta t) = \begin{pmatrix} \frac{1}{2} \Delta t^2 & \frac{1}{2} \Delta t^2 & \Delta t & \Delta t^2 \\ 1 & \frac{1}{2} \Delta t^2 & \Delta t & \Delta t^2 \\ 1 & 1 & \Delta t & \Delta t \\ 1 & 1 & 1 & \Delta t \end{pmatrix} \]

The covariance matrix \( Q \) is given by:
\[ Q_{k|k-1} = E \left[ w_k w_k^T \right] = q_v^2 \int_{t_0}^{t} d\tau \ \Gamma(\Delta t) \Gamma(\Delta t)^T \]

\[ = q_v^2 \begin{pmatrix} \frac{1}{20} \Delta t^5 & \frac{1}{20} \Delta t^5 & \frac{1}{8} \Delta t^4 & \frac{1}{8} \Delta t^4 & \frac{1}{6} \Delta t^3 & \frac{1}{6} \Delta t^3 & \frac{1}{5} \Delta t^2 & \frac{1}{5} \Delta t^2 \\ \frac{1}{20} \Delta t^5 & \frac{1}{20} \Delta t^5 & \frac{1}{8} \Delta t^4 & \frac{1}{8} \Delta t^4 & \frac{1}{6} \Delta t^3 & \frac{1}{6} \Delta t^3 & \frac{1}{5} \Delta t^2 & \frac{1}{5} \Delta t^2 \\ \frac{1}{8} \Delta t^4 & \frac{1}{8} \Delta t^4 & \frac{1}{3} \Delta t^3 & \frac{1}{3} \Delta t^3 & \frac{1}{2} \Delta t^2 & \frac{1}{2} \Delta t^2 & \Delta t & \Delta t \\ \frac{1}{6} \Delta t^3 & \frac{1}{6} \Delta t^3 & \frac{1}{2} \Delta t^2 & \frac{1}{2} \Delta t^2 & \Delta t & \Delta t & \Delta t & \Delta t \\ \frac{1}{6} \Delta t^3 & \frac{1}{6} \Delta t^3 & \frac{1}{2} \Delta t^2 & \frac{1}{2} \Delta t^2 & \Delta t & \Delta t & \Delta t & \Delta t \\ \frac{1}{6} \Delta t^3 & \frac{1}{6} \Delta t^3 & \frac{1}{2} \Delta t^2 & \frac{1}{2} \Delta t^2 & \Delta t & \Delta t & \Delta t & \Delta t \\ \frac{1}{6} \Delta t^3 & \frac{1}{6} \Delta t^3 & \frac{1}{2} \Delta t^2 & \frac{1}{2} \Delta t^2 & \Delta t & \Delta t & \Delta t & \Delta t \\ \frac{1}{6} \Delta t^3 & \frac{1}{6} \Delta t^3 & \frac{1}{2} \Delta t^2 & \frac{1}{2} \Delta t^2 & \Delta t & \Delta t & \Delta t & \Delta t \end{pmatrix} \]
The Coordinated Turn Model

Consider the state

\[
x = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix}
\]

and an object that is moving with constant speed a turn with constant angular velocity. This can be described by

\[
\begin{align*}
\ddot{x} &= -\Omega \dot{y} \\
\ddot{y} &= \Omega \dot{x}
\end{align*}
\]

we obtain the following system matrix

\[
A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\Omega \\ 0 & 0 & \Omega & 0 \end{pmatrix}
\]

\[
F(t_0, t) = \exp(\Delta t A)
\]

\[
F_{k|k-1} =
\begin{pmatrix}
1 & 0 & \sin \Omega \Delta t & -\frac{1-\cos \Omega \Delta t}{\Omega} \\
0 & 1 & 1-\cos \Omega \Delta t & \frac{\Omega}{\sin \Omega \Delta t} \\
0 & 0 & \cos \Omega \Delta t & -\sin \Omega \Delta t \\
0 & 0 & \sin \Omega \Delta t & \cos \Omega \Delta t
\end{pmatrix}
\]
How to determine Turn Rate

Assume constant velocity $v$. Then:

$$\Omega = \frac{a}{v} = \frac{\sqrt{\ddot{x}^2 + \ddot{y}^2}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

\(\ddot{x}\) and \(\ddot{y}\) can be approximated by

$$\ddot{x}(k) \approx \frac{\dot{x}(k) - \dot{x}(k - 1)}{\Delta t}$$