Bayesian Tracking: Basic Idea

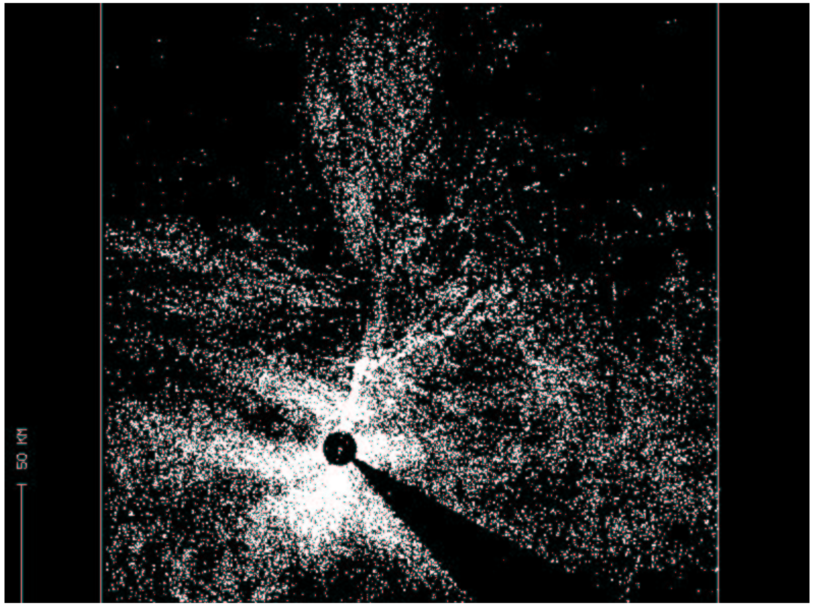
Iterative updating of conditional probability densities!

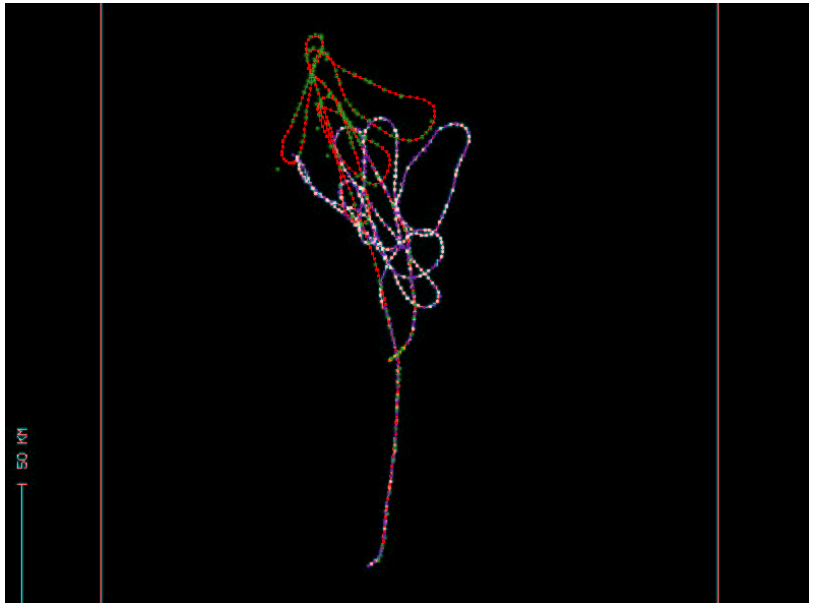
kinematic target state x_k at time t_k, **accumulated sensor data** $\mathcal{Z}^k$

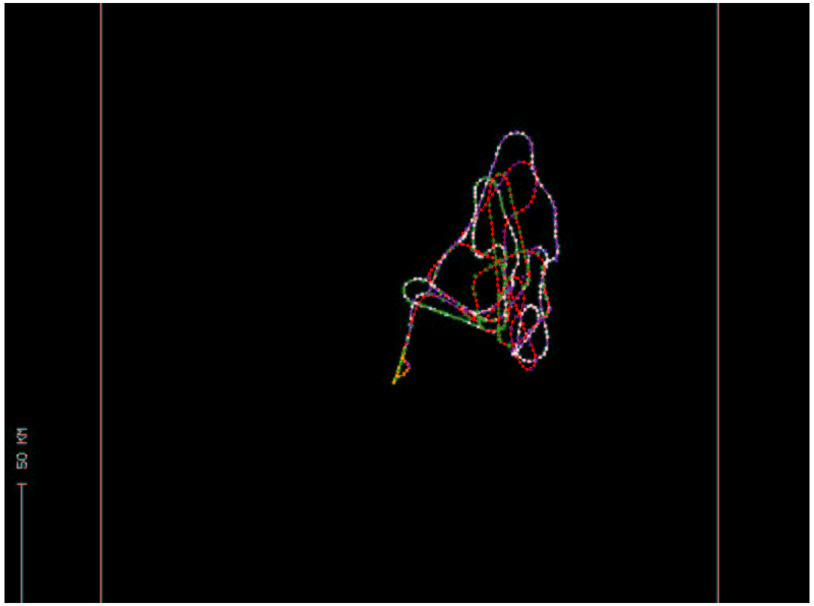
a priori knowledge: target dynamics models, sensor model, road maps

- **prediction:** $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \xrightarrow[\text{road maps}]{\text{dynamics model}} p(\mathbf{x}_k|\mathcal{Z}^{k-1})$
- **filtering:** $p(\mathrm{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor model}]{\text{sensor data } Z_k} p(\mathrm{x}_k|\mathcal{Z}^k)$
- **retrodiction:** $p(\mathbf{x}_{l-1}|\mathcal{Z}^k) \xleftarrow[\text{dynamics model}]{\text{filtering output}} p(\mathbf{x}_l|\mathcal{Z}^k)$

– ***finite mixture:*** **inherent ambiguity (data, model, road *network*)**
– ***optimal estimators:*** **e.g. minimum mean squared error (MMSE)**
– ***initiation of pdf iteration:*** **multiple hypothesis track extraction**







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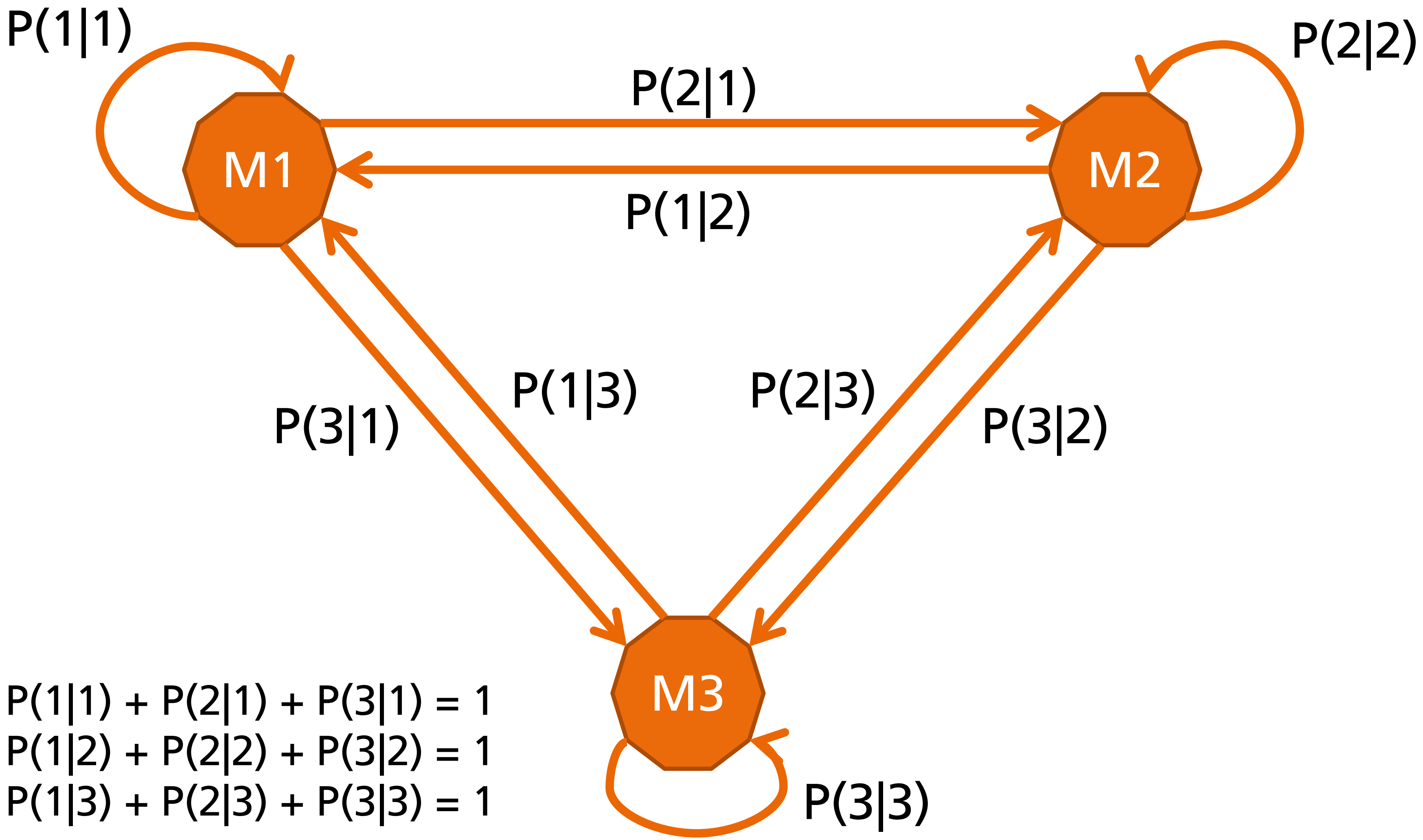
Quite general: agent switching between different modes of over-all behavior



P(1|1) + P(2|1) + P(3|1) = 1
P(1|2) + P(2|2) + P(3|2) = 1
P(1|3) + P(2|3) + P(3|3) = 1

A quite general mathematical structure: a graph, characterized by nodes (here: evolution models) and directed edges defining an adjacency matrix (here: transition matrix P, stochastic matrix: columns sum up to one)

initial information on which model is currently being in effect: $\mathrm{p}_k = (p_k^1, p_k^2, p_k^3)^\top$

Markov propagation: $$\mathrm{p}_k = \mathrm{P}\,\mathrm{p}_{k-1} = \begin{pmatrix} p(1|1) & p(1|2) & p(1|3) \\ p(2|1) & p(2|2) & p(2|3) \\ p(3|1) & p(3|2) & p(3|3) \end{pmatrix} \begin{pmatrix} p_{k-1}^1 \\ p_{k-1}^2 \\ p_{k-1}^3 \end{pmatrix}$$

Perron-Frobenius: the spectral radius of stochastic matrices is 1, 1 is also an eigenvalue and the corresponding eigenvector is positive.

Exercise: Consider the example: $$\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

and calculate the invariant state (eigenvector for eingenvalue 1). Show numerically or mathematically that each initial state converges to the invariant state.

IMM: Multiple Interacting Models j_k out of r alternatives at t_k

IMM dynamics: $$p(x_k, j_k | x_{k-1}, j_{k-1}) = \underbrace{p(j_k|j_{k-1})}_{\text{interaction}} \underbrace{\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

filtering at t_{k-1}: $$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\ \mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}^{j_{k-1}}_{k-1|k-1},\ \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),$$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r}\sum_{j_{k-1}=1}^{r} \int d\mathbf{x}_{k-1}\ \underbrace{p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1})}_{\text{IMM dynamics}}\ p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1})$$

$$= \sum_{j_k=1}^{r} \underbrace{\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}^{j_k j_{k-1}}_{k|k-1},\ \mathbf{P}^{j_k j_{k-1}}_{k|k-1}\big)}_{\approx p(j_k|\mathcal{Z}^{k-1})\ \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k-1}, \mathbf{P}^{j_k}_{k|k-1})}$$

with: $\mathbf{x}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}$, $\quad \mathbf{P}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}\mathbf{F}^{j_k\top}_{k|k+1} + \mathbf{D}^{j_k}_{k|k+1}$ (product formula)

Approximate GAUSSian mixture representation of $p(\mathbf{x}_k|\mathcal{Z}^{k-1})$ with r mixture components!

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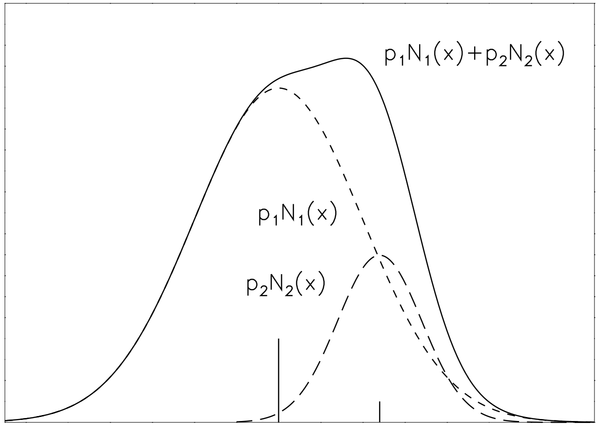
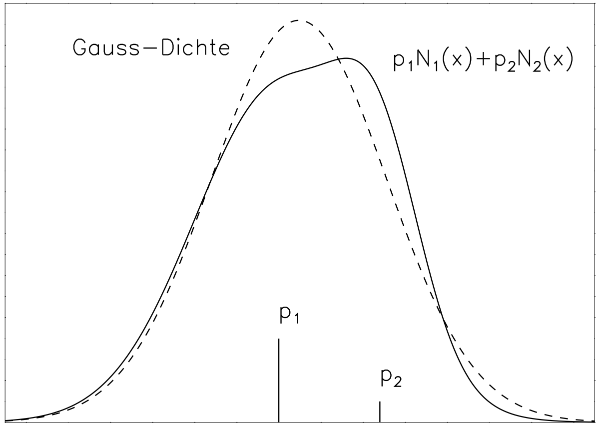
Moment Matching: Approximate an arbitrary pdf

$p(x)$ with $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{C}[x] = \mathbf{P}$ by $p(x) \approx \mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{P}\big)$!

here especially: $p(x) = \sum_i p_i \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i)$ (GAUSSian mixtures)





$$\mathbf{x} = \sum_i p_i \, \mathbf{x}_i$$

$$\mathbf{P} = \sum_i p_i \left\{ \mathbf{P}_i + \overbrace{(\mathbf{x}_i - \mathbf{x})(\mathbf{x}_i - \mathbf{x})^\top}^{\text{spread term}} \right\}$$

Bayesian filtering update based on IMM predictions ($r = 1$: KALMAN)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \sum_{j_k=1}^{r} \frac{p(j_k|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k-1},\, \mathbf{P}^{j_k}_{k|k-1}\big)}{\sum_{j'_k=1}^{r} p(j'_k|\mathcal{Z}^{k-1}) \int d\mathbf{x}_k\, \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j'_k}_{k|k-1},\, \mathbf{P}^{j'_k}_{k|k-1}\big)}$$

$$= \sum_{j_k=1}^{r} \frac{c_{j_k}\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k},\, \mathbf{P}^{j_k}_{k|k}\big)}{\sum_{j'_k=1}^{r} c_{j'_k}}$$ with: $c_{j_k} = p(j_k|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1},\, \mathbf{S}^{j_k}_{k|k}\big)$

$$= \sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^k)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k},\, \mathbf{P}^{j_k}_{k|k}\big)$$ (due to the product formula)

with: $p(j_k|\mathcal{Z}^k) = \frac{p(j_k|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{z}_k; \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1}, \mathbf{S}^{j_k}_{k|k}\big)}{\sum_{j'_k=1}^{r} p(j'_k|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{z}_k; \mathbf{H}_k\mathbf{x}^{j'_k}_{k|k-1}, \mathbf{S}^{j'_k}_{k|k}\big)}$ (mixture coefficients)

$$\mathbf{x}^{j_k}_{k|k} = \mathbf{x}^{j_k}_{k|k-1} + \mathbf{W}^{j_k}_{k|k}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1}), \qquad \mathbf{W}^{j_k}_{k|k} = \mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k^\top \mathbf{S}^{j_k\,-1}_{k|k} \qquad \text{(KALMAN)}$$

$$\mathbf{P}^{j_k}_{k|k} = \mathbf{P}^{j_k}_{k|k-1} - \mathbf{W}^{j_k}_{k|k-1}\mathbf{S}^{j_k}_{k|k}\mathbf{W}^{j_k}_{k|k-1}, \qquad \mathbf{S}^{j_k}_{k|k} = \mathbf{H}_k\mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k.$$

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IMM dynamics: $$p(x_k, j_k | x_{k-1}, j_{k-1}) = p(j_k | j_{k-1})\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}^{j_k}_{k|k-1}\big)$$

IMM prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r} \sum_{j_{k-1}=1}^{r} p_{j_k j_{k-1}}\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}^{j_k j_{k-1}}_{k|k-1},\ \mathbf{P}^{j_k j_{k-1}}_{k|k-1}\big)$$

$$p_{j_k j_{k-1}} = p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1}), \qquad \mathbf{x}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}, \qquad \mathbf{P}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}\mathbf{F}^{j_k\top}_{k|k+1} + \mathbf{D}^{j_k}_{k|k+1}$$

IMM mixing step: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k-1})\ \mathcal{N}(\mathbf{x}_k;\ \mathbf{x}^{j_k}_{k|k-1},\ \mathbf{P}^{j_k}_{k|k-1})$$

$$p(j_k|\mathcal{Z}^{k-1}) = \sum\nolimits_{j_{k-1}=1}^{r} p_{j_k j_{k-1}}, \qquad \mathbf{x}^{j_k}_{k|k-1} = \frac{1}{p(j_k|\mathcal{Z}^{k-1})} \sum\nolimits_{j_{k-1}=1}^{r} p_{j_k j_{k-1}} \mathbf{x}^{j_k j_{k-1}}_{k|k-1}$$

$$\mathbf{P}^{j_k}_{k|k-1} = \frac{1}{p(j_k|\mathcal{Z}^{k-1})} \sum\nolimits_{j_{k-1}=1}^{r} p_{j_k j_{k-1}} \Big(\mathbf{P}^{j_k j_{k-1}}_{k|k-1} + (\mathbf{x}^{j_k j_{k-1}}_{k|k-1} - \mathbf{x}^{j_k}_{k|k-1})(\cdots)^\top\Big)$$

IMM filtering: $$p(\mathbf{x}_k|\mathcal{Z}^{k}) = \sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k})\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}^{j_k}_{k|k},\ \mathbf{P}^{j_k}_{k|k}\big)$$

$$p(j_k|\mathcal{Z}^{k}) = \frac{p(j_k|\mathcal{Z}^{k-1})\ \mathcal{N}\Big(\mathbf{z}_k;\ \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1},\ \mathbf{S}^{j_k}_{k|k}\Big)}{\sum_{j'_k=1}^{r} p(j'_k|\mathcal{Z}^{k-1})\ \mathcal{N}\Big(\mathbf{z}_k;\ \mathbf{H}_k\mathbf{x}^{j'_k}_{k|k-1},\ \mathbf{S}^{j'_k}_{k|k}\Big)}, \qquad \mathbf{S}^{j_k}_{k|k} = \mathbf{H}_k\mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k$$

$$\mathbf{x}^{j_k}_{k|k} = \mathbf{x}^{j_k}_{k|k-1} + \mathbf{W}^{j_k}_{k|k}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1}), \qquad \mathbf{P}^{j_k}_{k|k} = \mathbf{P}^{j_k}_{k|k-1} - \mathbf{W}^{j_k}_{k|k-1}\mathbf{S}^{j_k}_{k|k}\mathbf{W}^{j_k}_{k|k-1}, \qquad \mathbf{W}^{j_k}_{k|k} = \mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k^\top \mathbf{S}^{j_k^{-1}}_{k|k}$$

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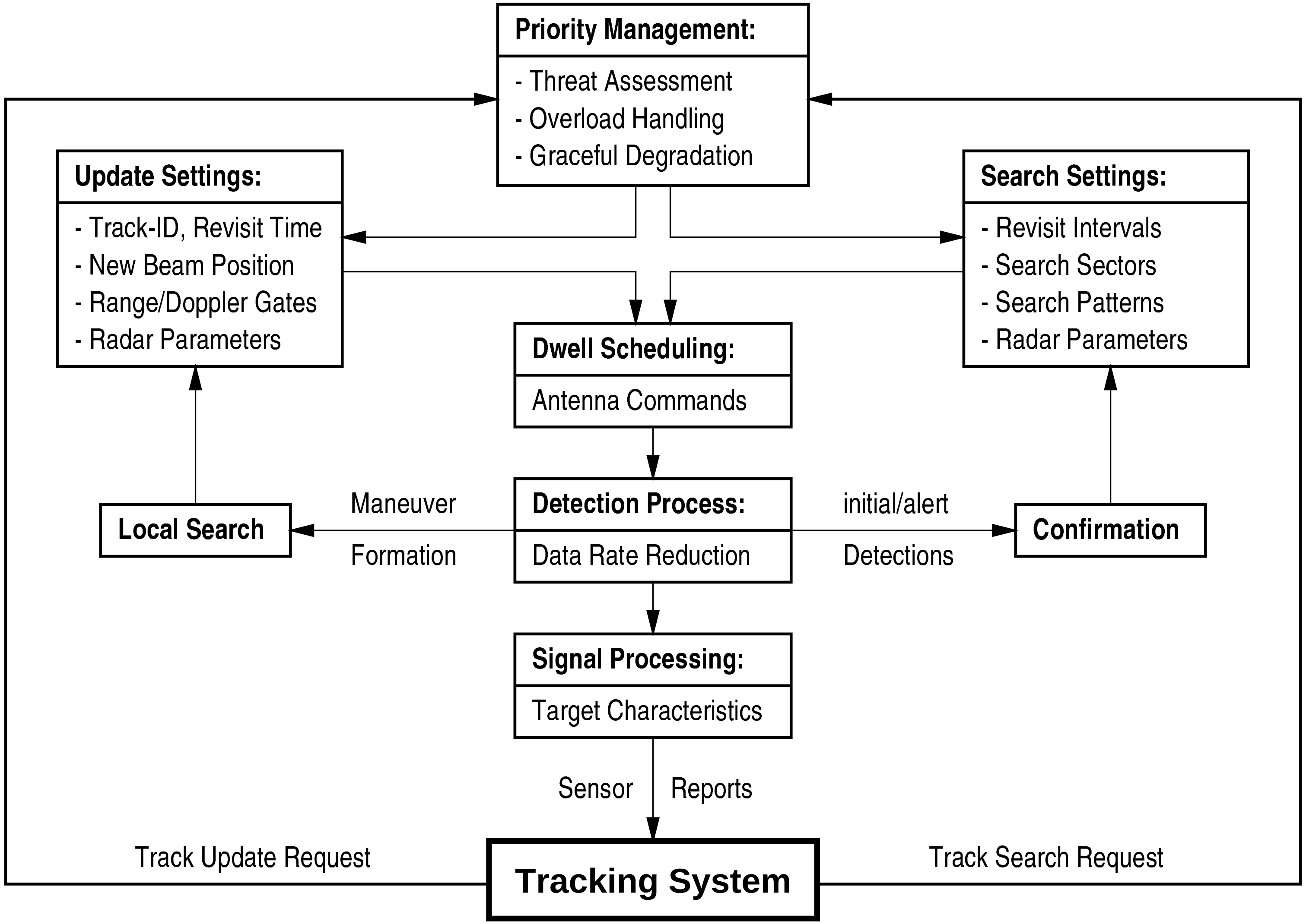
Radar Function Control: Information Flow



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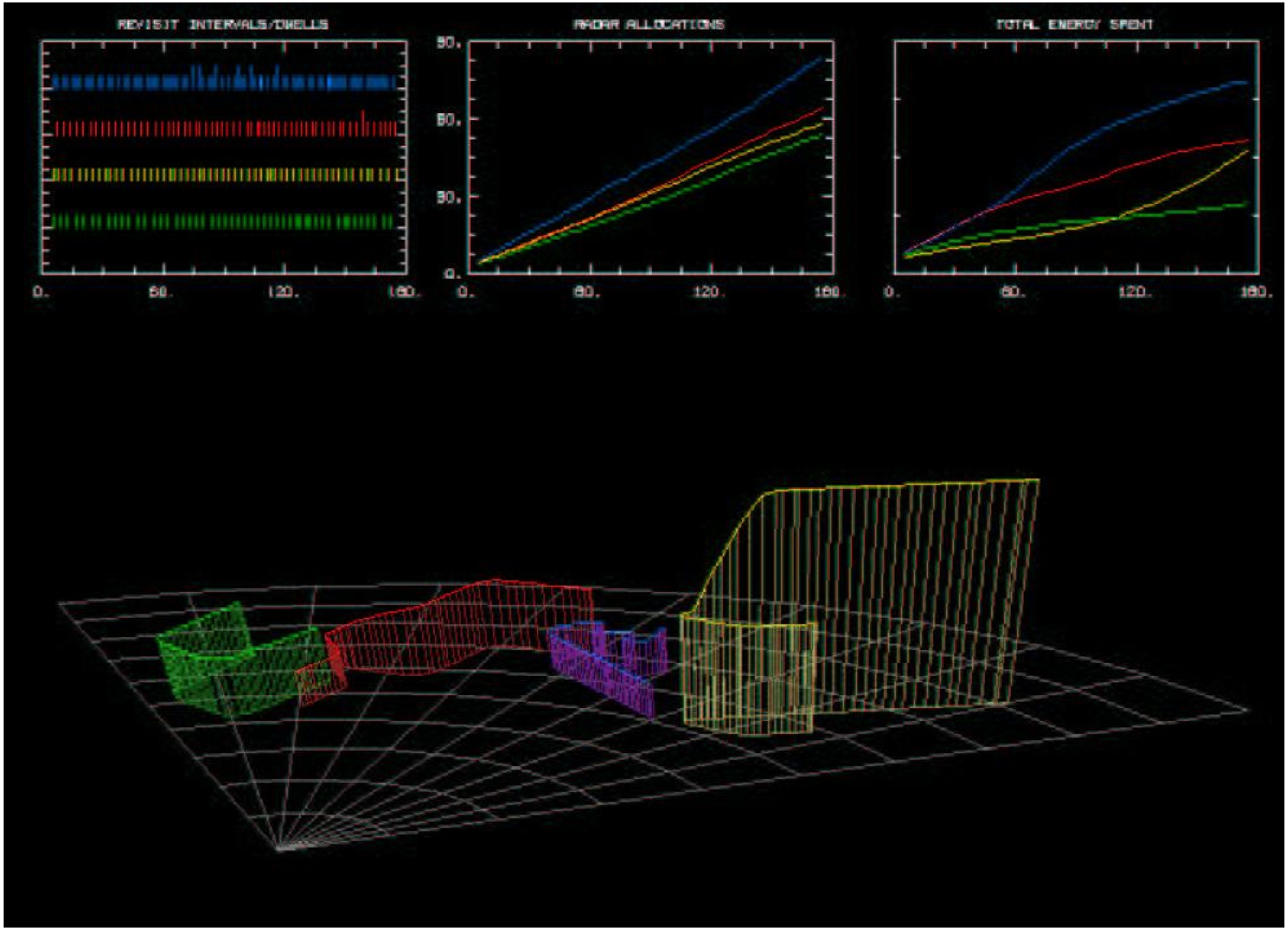
Phased-Array Tracking: Standard Sensor Management



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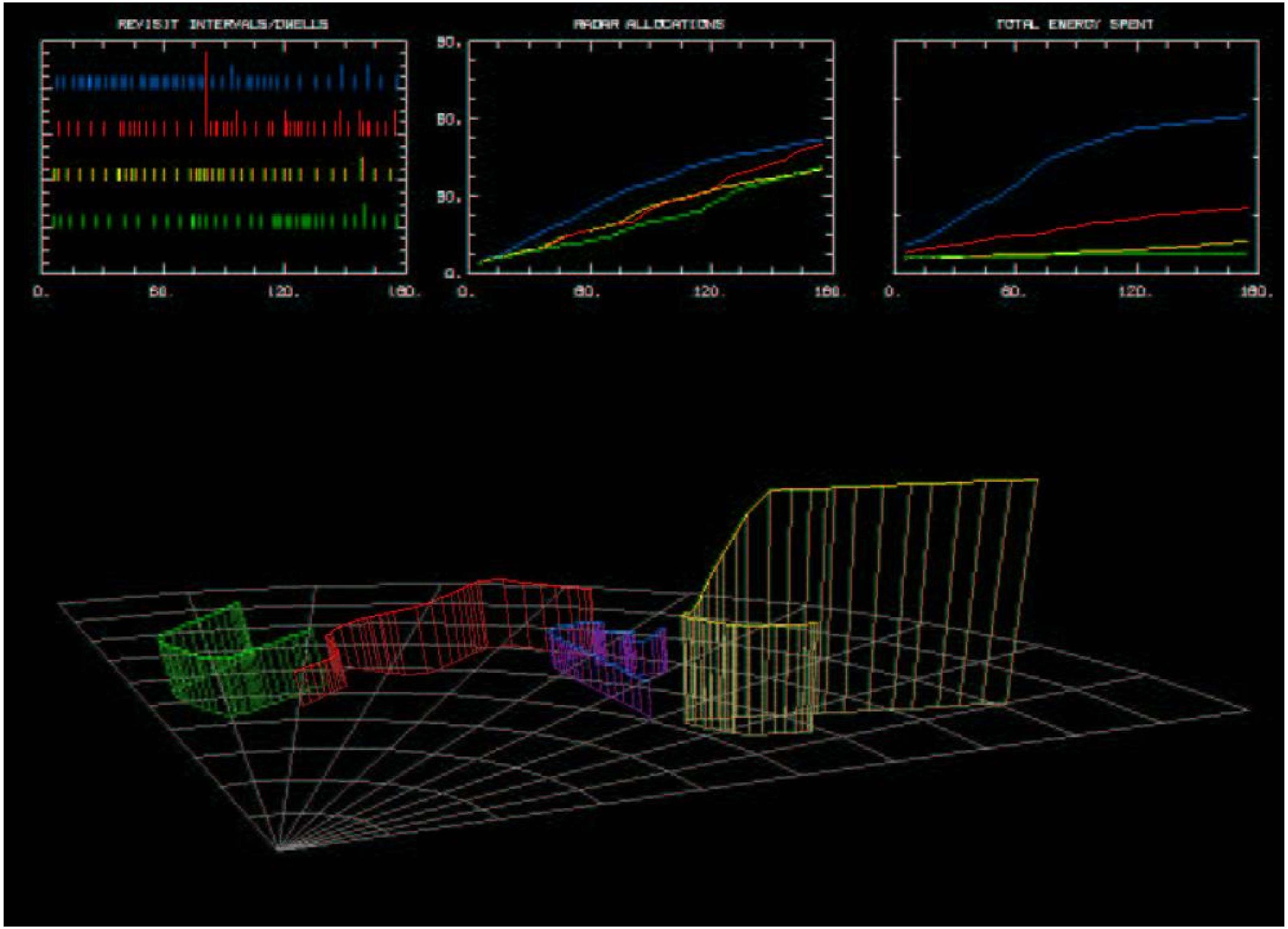
Phased-Array Tracking: Adaptive Sensor Management



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Phased Array Radar: Adaptive Local Search

sensor resources allocated by the tracker

- adaptive revisit time control (lack of information)
- selective radar beam positioning (state prediction)
- if necessary: locally optimal search ($\rightarrow$ BAYES)

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a simple radar beam and detection model

- beam positioning error: $\Delta \mathbf{b}_k^2 = |\mathbf{x}_k - \mathbf{b}_k|^2 / B^2$

  $\mathbf{x}_k$: target direction, $\mathbf{b}_k$: current beam position, B: radar beam width

- signal-to-noise ratio: $s(\mathbf{x}_k; \mathbf{b}_k) = s_0 \, \mathrm{e}^{-\log 2 \Delta \mathbf{b}_k^2}$
- false alarm probability: P_F — SWERLING model
- detection probability: $P_D(\mathbf{x}_k; \mathbf{b}_k) = P_F^{\frac{1}{1+s(\mathbf{x}_k;\mathbf{b}_k)}}$

BAYESian Beam Positioning and Local Search

1. first dwell $\rightarrow$ point radar beam to predicted target position $\rightarrow \mathbf{b}_k^1$
2. successful detection $\rightarrow$ filtering/prediction/revisit time $\rightarrow$ GOTO 1!

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3. no success $\rightarrow$ Bayesian processing of this 'negative' output ($\neg \mathrm{D}_1$):

$$\boxed{p(\mathbf{x}_k|\neg \mathrm{D}_1, \mathcal{Z}^{k-1}) \propto \left[1 - P_D(\mathbf{x}_k; \mathbf{b}_k^1)\right] p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

4. next dwell $\rightarrow$ point beam to the maximum of this conditional pdf!

BAYESian Beam Positioning and Local Search

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4. next dwell $\rightarrow$ point beam to the maximum of this conditional pdf!
5. successful detection $\rightarrow$ filtering/prediction/revisit time $\rightarrow$ GOTO 1!
6. no success $\rightarrow$ calculate: $p(u_k, v_k|\neg\mathrm{D}_1, \neg\mathrm{D}_2, \mathcal{Z}^{k-1}) \rightarrow$ GOTO 4!

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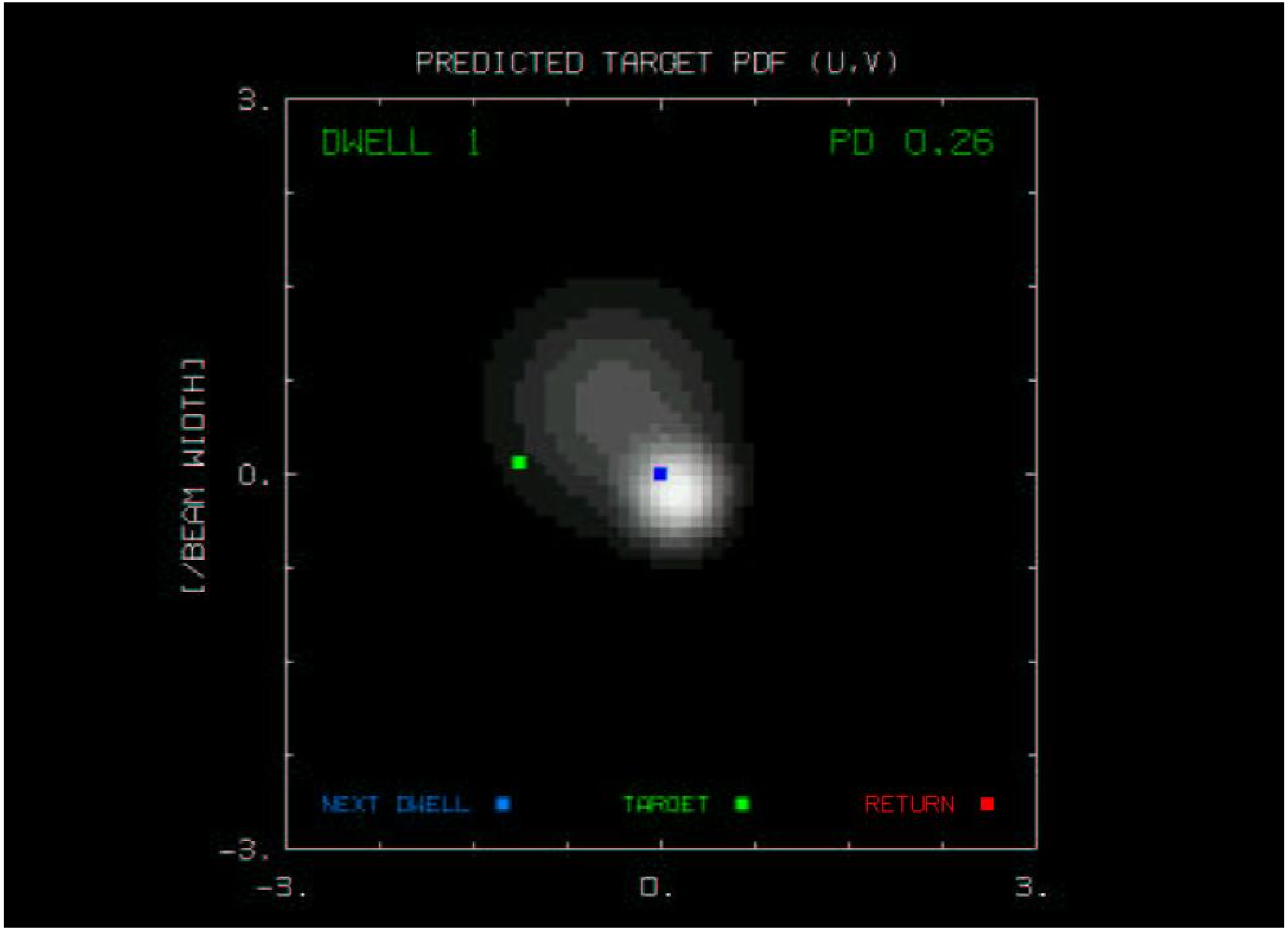
BAYESian local search: example with an IMM pdf



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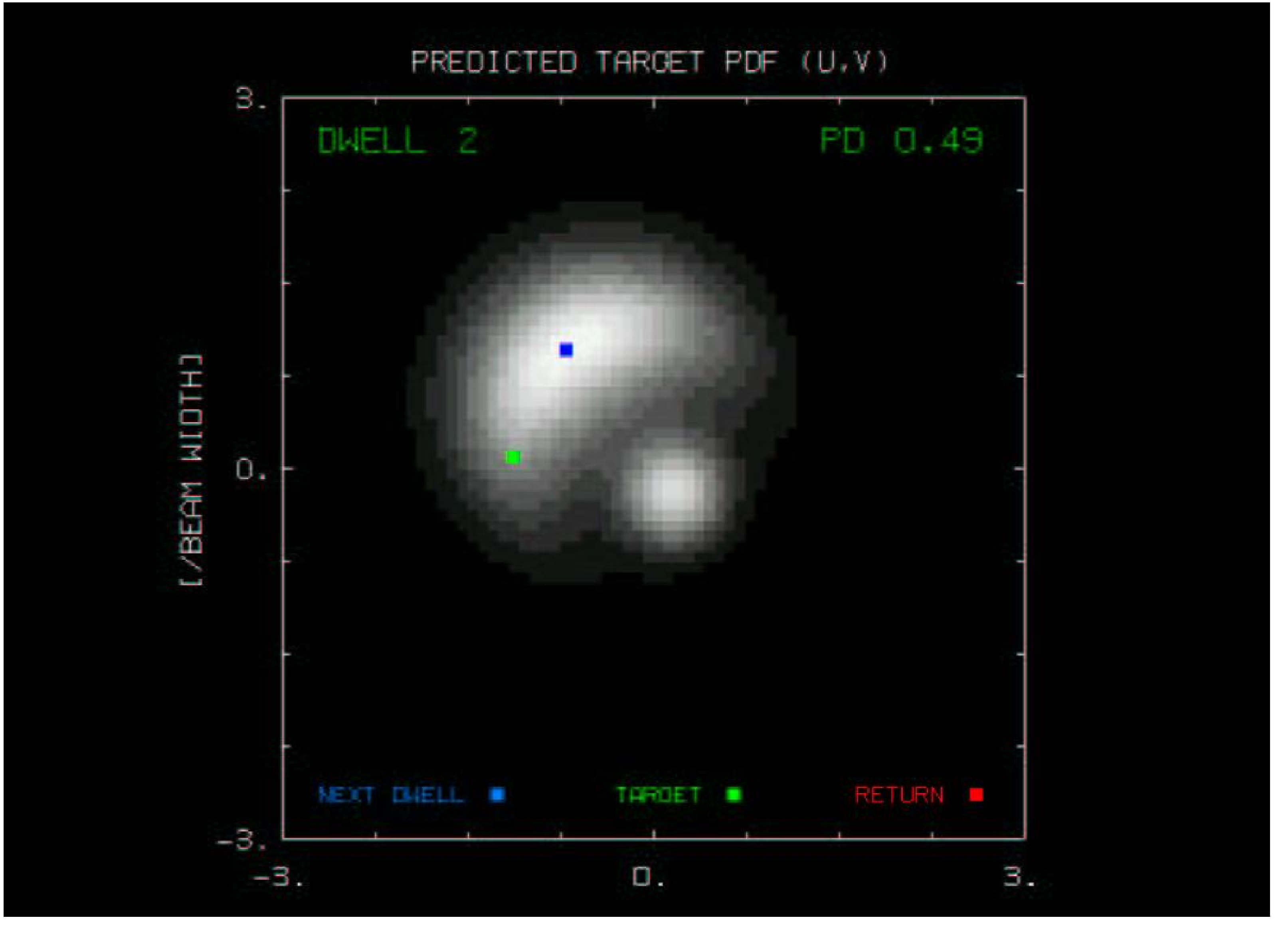

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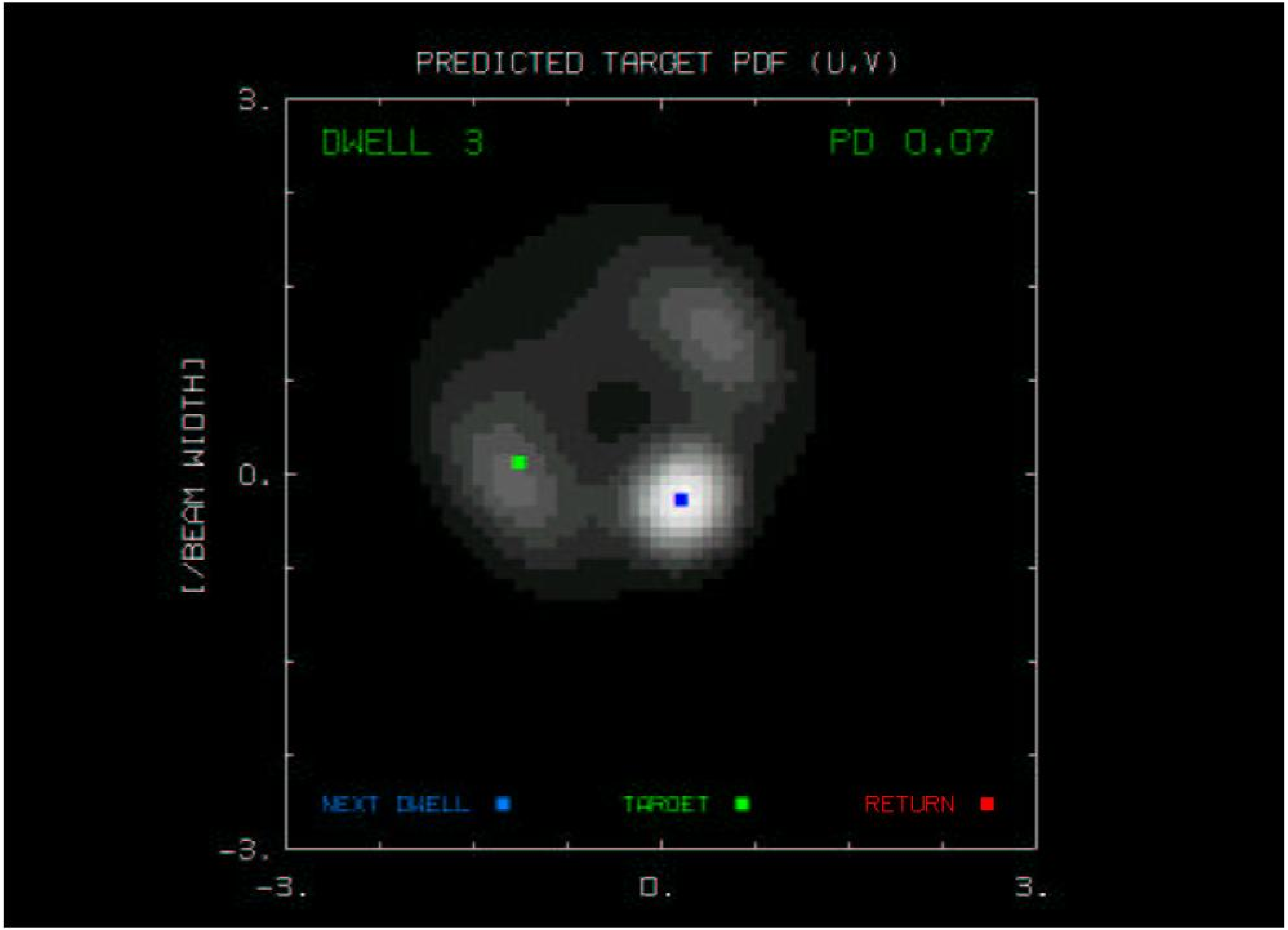

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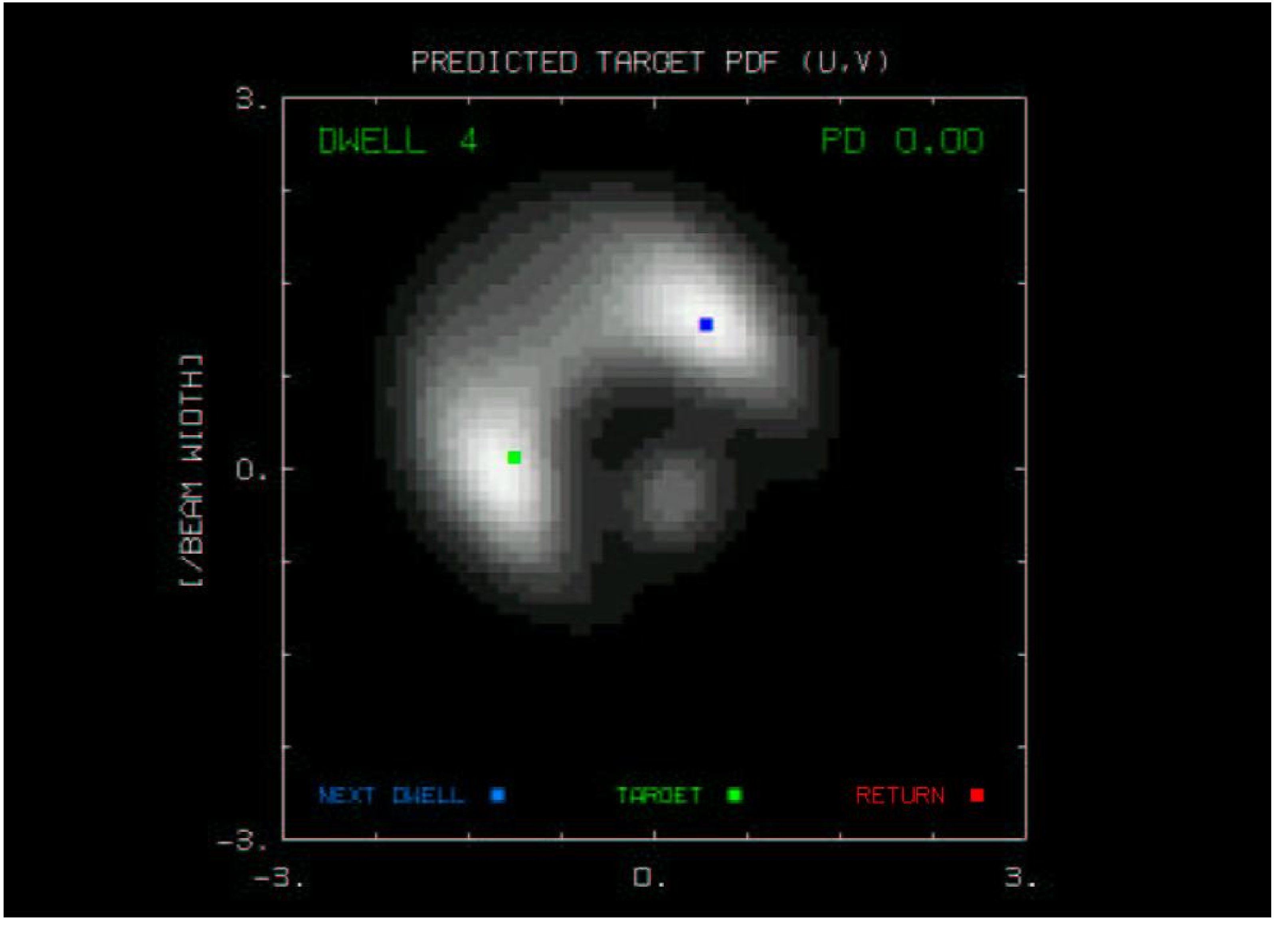

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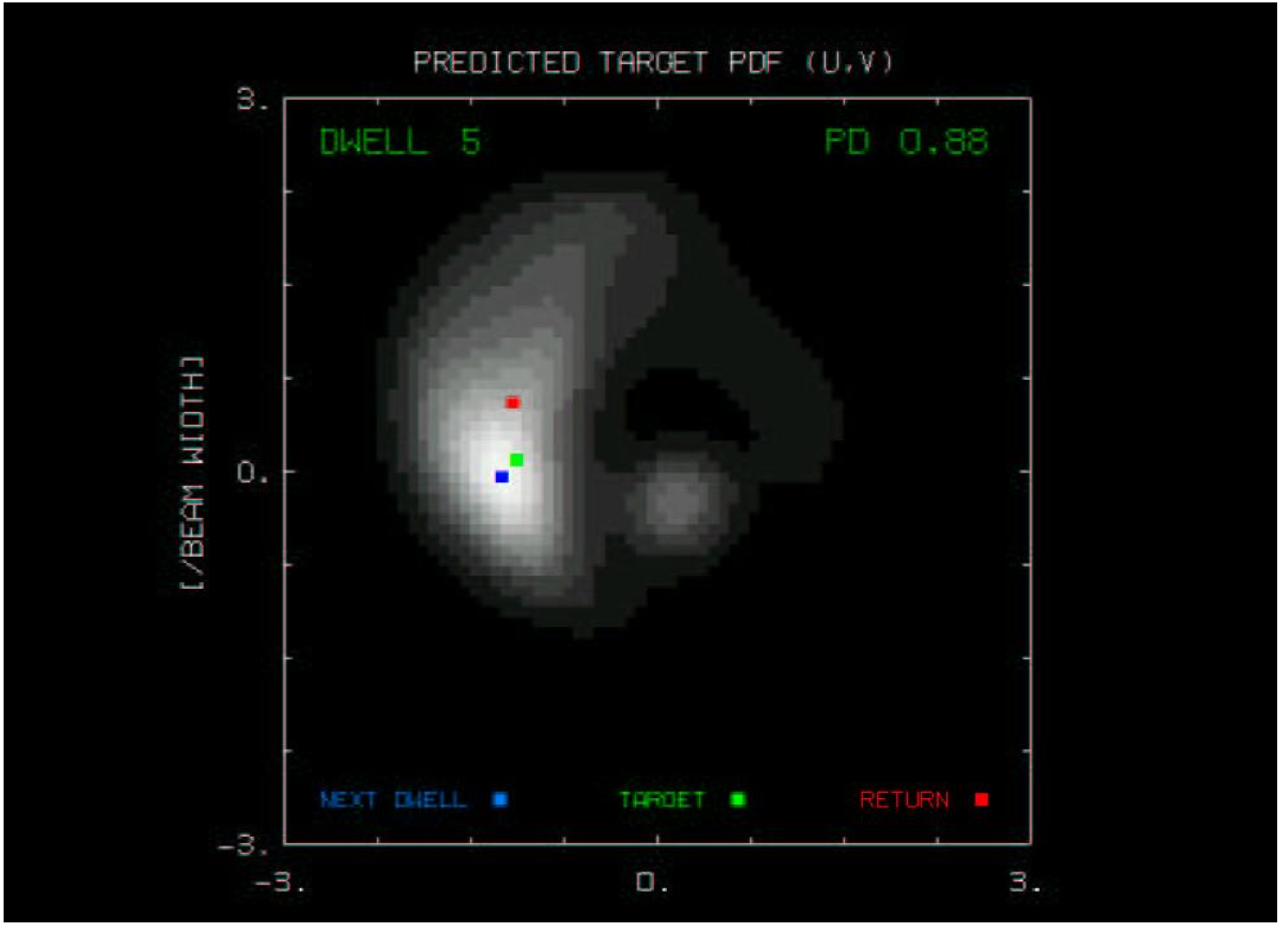

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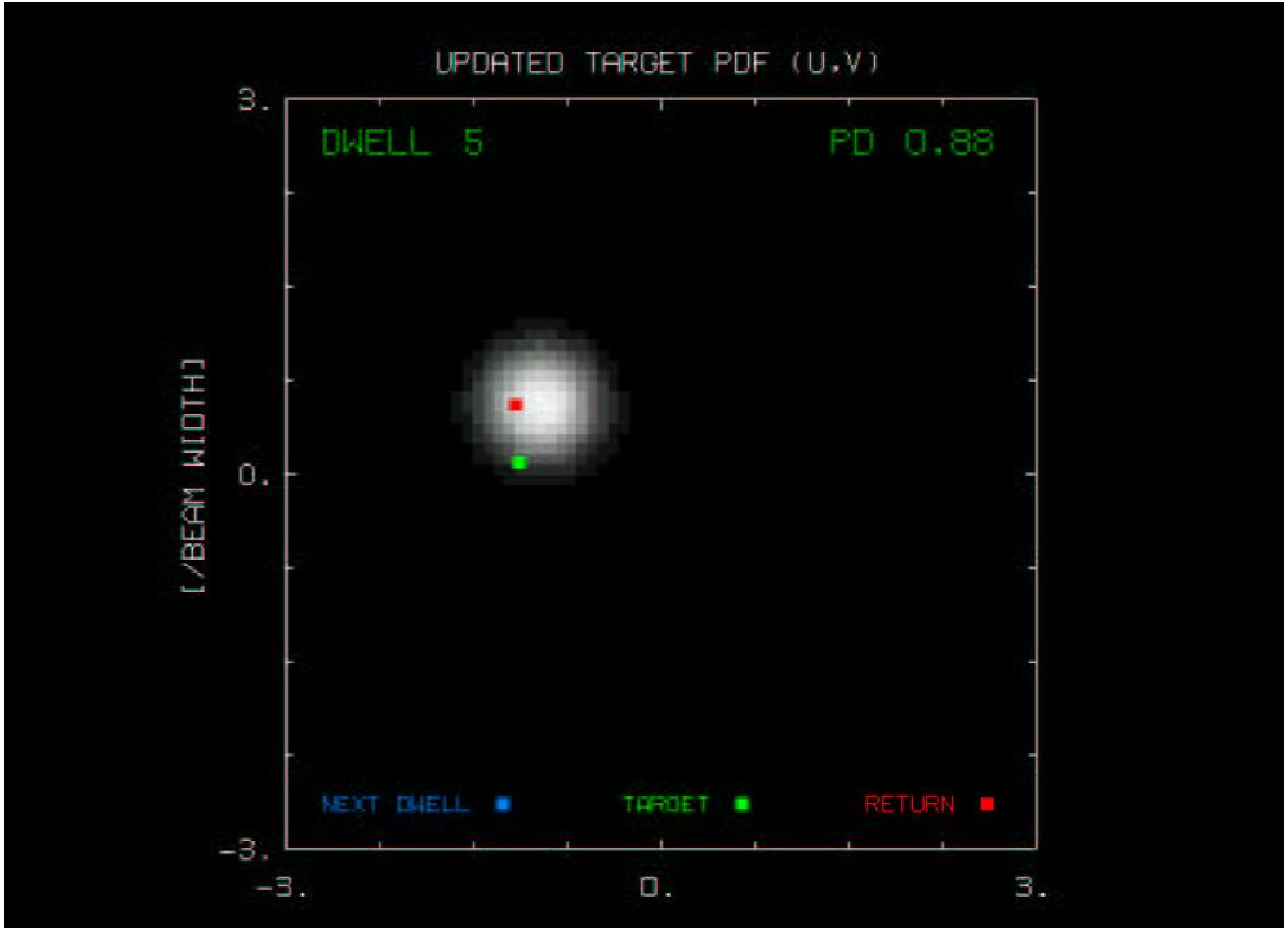

A *'negative' sensor output* - i.e. missing, but expected data - provides information on the target position.

- **direct impact on sensor management**
- **prerequisite: appropriate sensor model**

IMM Models: Retrodiction

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k)$$ (marginal density!)

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$$= \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}\mathcal{Z}^k)\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction for } t_{l+1}}$$

for time $l+1$ assume: $p(\mathbf{x}_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1}}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}^k_{i_{l+1}}, \mathbf{P}^k_{i_{l+1}}\big)$

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$$p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^k}) = p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^l}) = \frac{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)\, p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}{\sum_{i_l} \int \mathbf{x}_l\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)}_{\text{IMM model}}\, \underbrace{p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}_{\text{filtering } t_l}}$$

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IMM Models: Retrodiction

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$$p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^k}) = p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^l}) = \frac{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)\, p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}{\sum_{i_l} \int \mathbf{x}_l\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)}_{\text{IMM model}}\, \underbrace{p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}_{\text{filtering } t_l}}$$

$$= \mathbf{c}_{i_{l+1}i_l}(\mathbf{x}_{l+1})\, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}_{i_l} + \mathbf{W}_{i_{l+1}i_l}\big(\mathbf{x}_{l+1} - \mathbf{x}_{i_{l+1}i_l}\big),\, \mathbf{P}_{i_l} - \mathbf{W}_{i_{l+1}i_l}\mathbf{P}_{i_{l+1}i_l}\mathbf{W}_{i_{l+1}i_l}{}^{\top}\Big) \qquad \textbf{product formula!}$$

with: $\mathbf{c}_{i_{l+1}i_l}(\mathbf{x}_{l+1}) = \dfrac{\boldsymbol{\mu}_{i_{l+1}i_l}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{i_{l+1}i_l},\, \mathbf{P}_{i_{l+1}i_l}\big)}{\sum_{i_l} \boldsymbol{\mu}_{i_{l+1}i_l}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{i_{l+1}i_l},\, \mathbf{P}_{i_{l+1}i_l}\big)}$ $\qquad \mathbf{W}_{i_{l+1}i_l} = \mathbf{P}_{i_l}\mathbf{F}^{\top}_{i_{l+1}}(\mathbf{F}_{i_{l+1}}\mathbf{P}_{i_l}\mathbf{F}^{\top}_{i_{l+1}} + \mathbf{D}_{i_{l+1}})^{-1}$

$$\approx \mathbf{c}_{i_{l+1}i_l}(\mathbf{x}^k_{i_{l+1}})$$

Fraunhofer FKIE

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

$$= \sum_{i_{l+1}, i_l} \boldsymbol{\mu}^k_{i_{l+1} i_l} \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_{l+1} i_l},\, \mathbf{P}^k_{i_{l+1} i_l}\Big)$$

exponential growth of dynamics histories $i_{i_{l+1} i_l \dots}$!

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

$$= \sum_{i_{l+1}, i_l} \boldsymbol{\mu}^k_{i_{l+1} i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_{l+1} i_l},\, \mathbf{P}^k_{i_{l+1} i_l}\Big)$$

exponential growth of dynamics histories $i_{i_{l+1} i_l \dots}$!

$$= \sum_{i_l} \underbrace{\sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1} i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_{l+1} i_l},\, \mathbf{P}^k_{i_{l+1} i_l}\Big)}_{\text{approximation: moment matching!}}$$

finally: $p(\mathbf{x}_l|\mathcal{Z}^k) \approx \sum_{i_l} \boldsymbol{\mu}^k_{i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_l},\, \mathbf{P}^k_{i_l}\Big)$

generalize: model histories of *variable* length!

IMM Modeling: Suboptimal Realization

- **Conventional KALMAN filtering**

  Only *one* component: worst-case assumtion

- **standard IMM filter (as discussed!)**

  Approximate after prediction, *before* update by r components! Effort: $\sim r$ KALMAN filter

- **GPB: Generalized Pseudo-BAYESian**

  Approximate *after* measurement processing by r components! Effort: $\sim r^2$ KALMAN filter

- **IMM-MHT filter (nearly optimal)**

  Accept longer dynamics histories $\rightarrow$ *variable* number of components!

Extendable to ambiguity with respect to sensor models!

Improved Approximation: Simple Approach

Consider *dynamics histories* of length κ: $\mathbf{i}_k = (i_k, i_{k-1}, \ldots, i_{k-\kappa+1})$

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als Filterung:

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prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{i_k, \mathbf{i}_{k-1}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k, \mathbf{x}_{k-1}, \mathbf{i}_{k-1}|\mathcal{Z}^{k-1})$$

Improved Approximation: Simple Approach

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Prďž˝diktion:

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{i_k, \mathbf{i}_{k-1}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k, \mathbf{x}_{k-1}, \mathbf{i}_{k-1}|\mathcal{Z}^{k-1})$$

$$= \underbrace{\sum_{i_k, \dots, i_{k-\kappa}}}_{\kappa+1 \text{ Summen}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k|\mathbf{x}_{k-1}, i_{k-1})\, p(\mathbf{x}_{k-1}, i_{k-1}, \dots, i_{k-\kappa}|\mathcal{Z}^{k-1})$$

Improved Approximation: Simple Approach

Consider *dynamics histories* of length κ: $\mathbf{i}_k = (i_k, i_{k-1}, \dots, i_{k-\kappa+1})$

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prediction:

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$$= \underbrace{\sum_{i_k, \dots, i_{k-\kappa}}}_{\kappa+1 \text{ sums}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k|\mathbf{x}_{k-1}, i_{k-1})\, p(\mathbf{x}_{k-1}, i_{k-1}, \dots, i_{k-\kappa}|\mathcal{Z}^{k-1})$$

$$= \sum_{i_k, \dots, i_{k-\kappa}} \boldsymbol{\mu}_{i_k, \dots, i_{k-\kappa}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{i_k, \dots, i_{k-\kappa}},\, \mathbf{P}_{i_k, \dots, i_{k-\kappa}}\big) = \sum_{\mathbf{i}_k} \underbrace{\sum_{i_{k-\kappa}} \boldsymbol{\mu}_{\mathbf{i}_k, i_{k-\kappa}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{\mathbf{i}_k, i_{k-\kappa}},\, \mathbf{P}_{\mathbf{i}_k, i_{k-\kappa}}\big)}_{\text{second order approximation!}}$$

Fraunhofer FKIE

Improved Approximation: Simple Approach

Consider *dynamics histories* of length κ: $\mathbf{i}_k = (i_k, i_{k-1}, \dots, i_{k-\kappa+1})$

as filtering:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \underbrace{\sum_{\mathbf{i}_{k-1}}}_{\kappa \text{ sums}} \boldsymbol{\mu}_{\mathbf{i}_{k-1}} \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{\mathbf{i}_{k-1}},\, \mathbf{P}_{\mathbf{i}_{k-1}}\big)$$

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finally: $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \sum_{\mathbf{i}_k} \boldsymbol{\mu}^*_{\mathbf{i}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^*_{\mathbf{i}_k},\, \mathbf{P}^*_{\mathbf{i}_{k-1}}\big)$ (for retrodiction analogous!)

Design of IMM Modelling

- ***number* r *of models:*** **relevant only for standard IMM**
- ***decisive:*** **sufficiently many Gausßian picture components**
- ***irrelevant:*** **by r or length of dynamics khistories n_H**
- ***recommendation:*** **worst/best case, histories ($r = 2$, $n_H = 3$)**
- ***benefit:*** **interpretable, close-to-reality dynamics parameters**

Demonstration