Bayesian Tracking: Basic Idea

Iterative updating of conditional probability densities!

kinematic target state x_k at time t_k, **accumulated sensor data** $\mathcal{Z}^k$

a priori knowledge: target dynamics models, sensor model, road maps

- **prediction:** $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \xrightarrow[\text{road maps}]{\text{dynamics model}} p(\mathbf{x}_k|\mathcal{Z}^{k-1})$
- **filtering:** $p(\mathrm{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor model}]{\text{sensor data } Z_k} p(\mathrm{x}_k|\mathcal{Z}^k)$
- **retrodiction:** $p(\mathbf{x}_{l-1}|\mathcal{Z}^k) \xleftarrow[\text{dynamics model}]{\text{filtering output}} p(\mathbf{x}_l|\mathcal{Z}^k)$

– ***finite mixture:*** **inherent ambiguity (data, model, road *network*)**
– ***optimal estimators:*** **e.g. minimum mean squared error (MMSE)**
– ***initiation of pdf iteration:*** **multiple hypothesis track extraction**

- $p(\mathbf{x}_k|\mathcal{Z}^{k-1})$ is a *prediction* of the target state at time t_k based on all measurements in the *past*.

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, \mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \quad \text{marginal pdf}$$

$$= \int d\mathbf{x}_{k-1}\, \underbrace{p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1})}_{\text{object dynamics!}}\, \underbrace{p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}_{\text{idea: iteration!}} \quad \text{notion of a conditional pdf}$$

often: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1})$ (MARKOV)

sometimes: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}}_{\text{deterministic}}\mathbf{x}_{k-1},\, \underbrace{\mathbf{D}_{k|k-1}}_{\text{random}}\big)$ (linear GAUSS-MARKOV)

- $p(Z_k, m_k|\mathbf{x}_k) \propto \ell(\mathbf{x}_k; Z_k, m_k)$ describes, what the *current* sensor output Z_k, m_k can say about the current target state $\mathbf{x}_k$ and is called *likelihood function*.

sometimes: $\ell(\mathbf{z}_k; \mathbf{x}_k) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$ (1 target, 1 measurement)

iteration formula: $$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k) \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k) \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}$$

GAUSSian transition pdf: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1})$

with: $\underbrace{\mathbf{F}_{k|k-1} \text{ (evolution matrix)},}_{\text{describes deterministic motion}}$ $\underbrace{\mathbf{D}_{k|k-1} \text{ (dynamics covariance matrix)}}_{\text{models of random maneuvers}}$

GAUSSian posterior: $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1})$

$$
\begin{aligned}
p(\mathbf{x}_k|\mathcal{Z}^{k-1}) &= \int d\mathbf{x}_{k-1}\ \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)}_{\text{dynamics model}}\ \underbrace{\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big)}_{\text{posterior at time } t_{k-1}} \\
&= \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}}_{=:\mathbf{x}_{k|k-1}},\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}^{\top} + \mathbf{D}_{k|k-1}}_{=:\mathbf{P}_{k|k-1}}\big) \\
&\quad \times \underbrace{\int d\mathbf{x}_{k-1}\ \mathcal{N}\big(\mathbf{x}_{k-1};\, \ldots,\, \ldots\big)}_{=1 \quad \text{(normalization!)}} \qquad \text{(exploit product formula!)} \\
&= \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})
\end{aligned}
$$

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0}, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

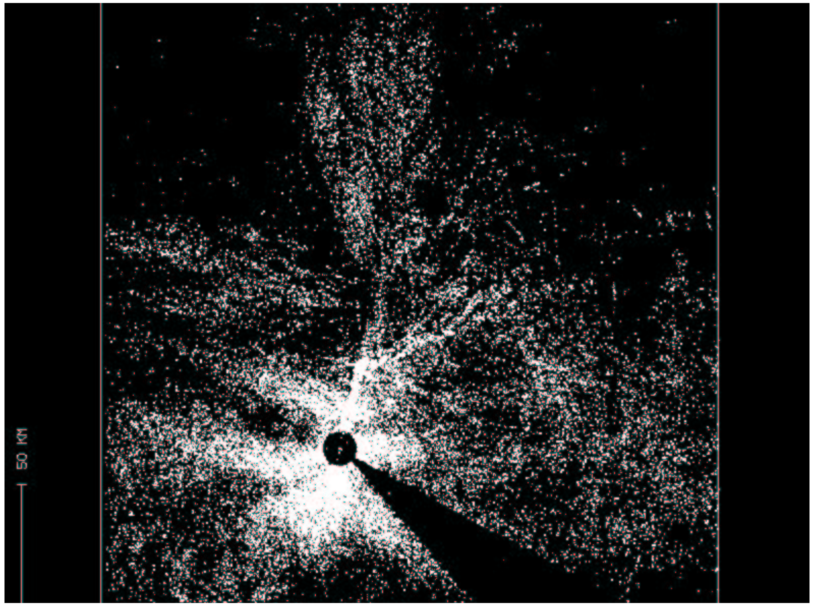
filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k}, \mathbf{P}_{k|k})$

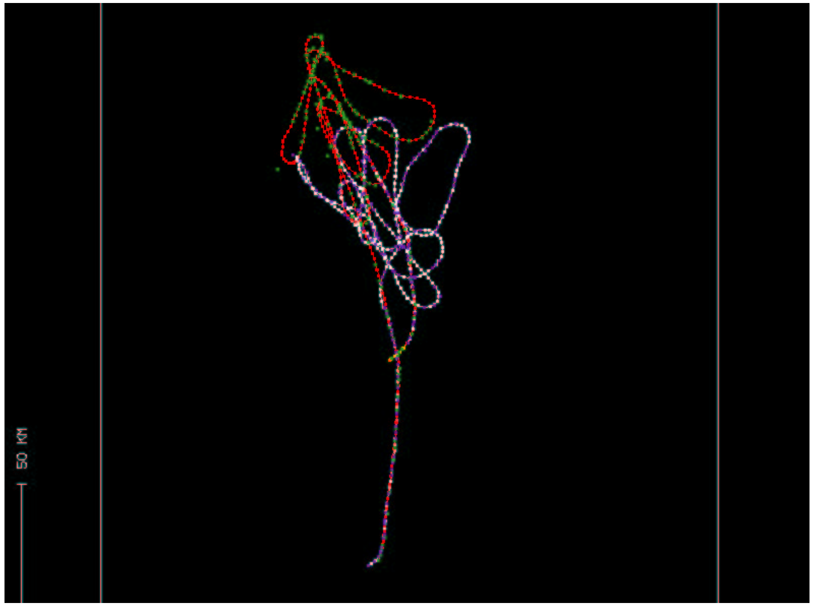
$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

retrodiction: $\mathcal{N}(\mathbf{x}_l;\, \mathbf{x}_{l|k}, \mathbf{P}_{l|k}) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k}, \mathbf{P}_{l+1|k})$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$


50 KM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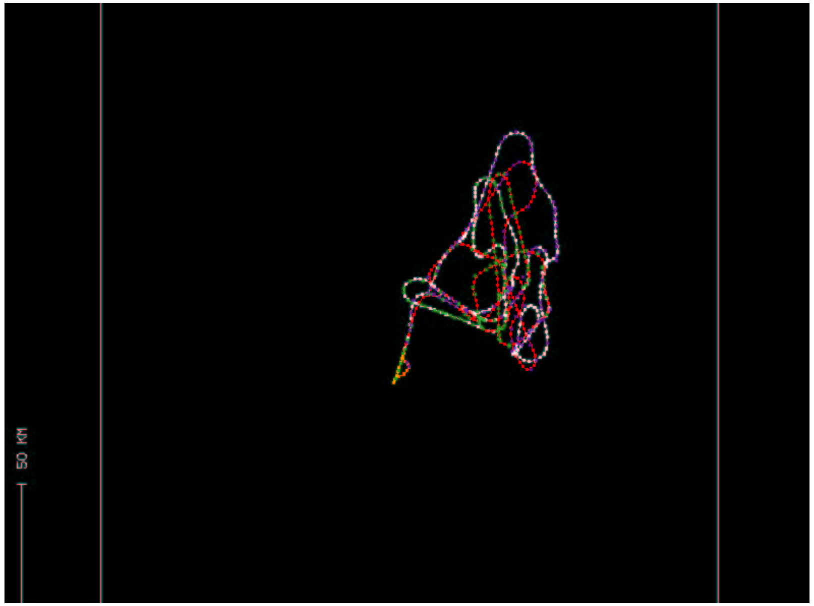
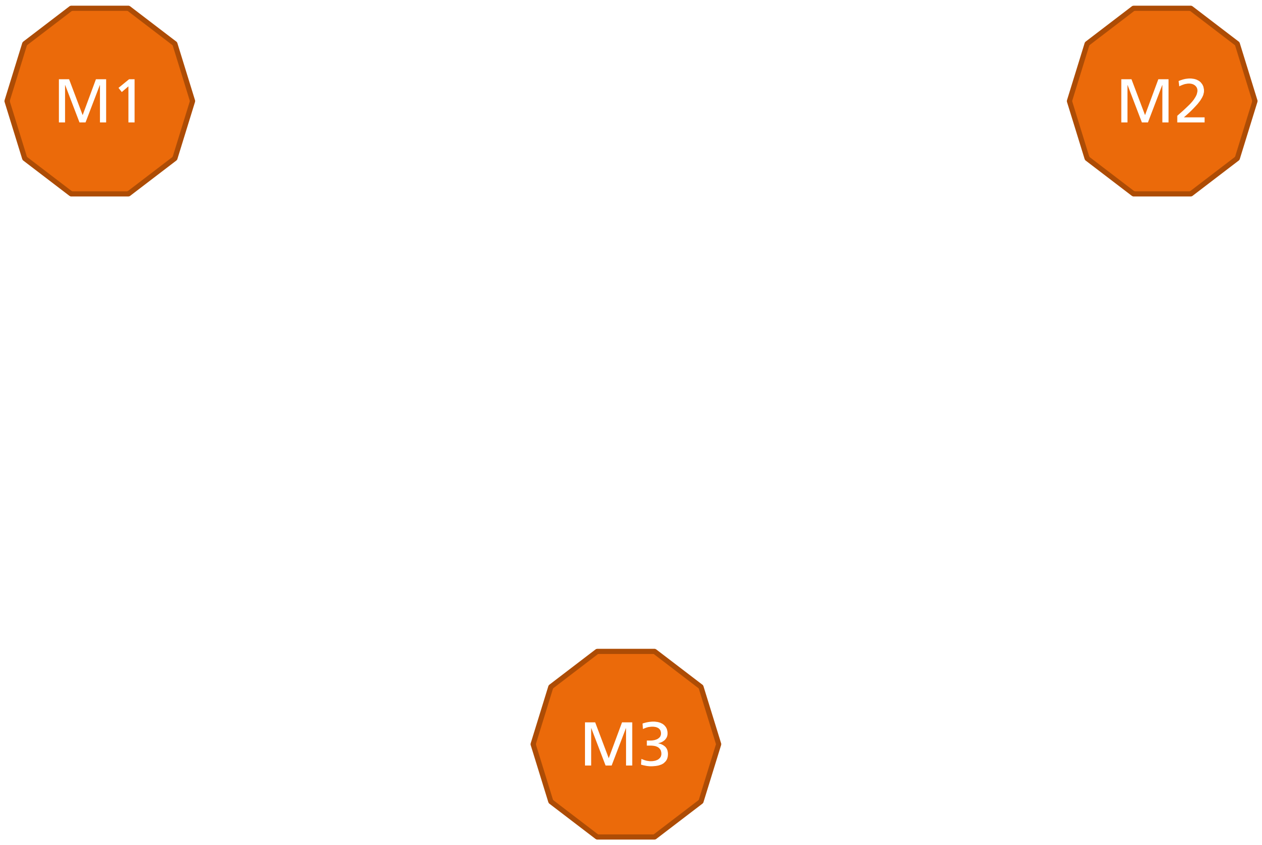
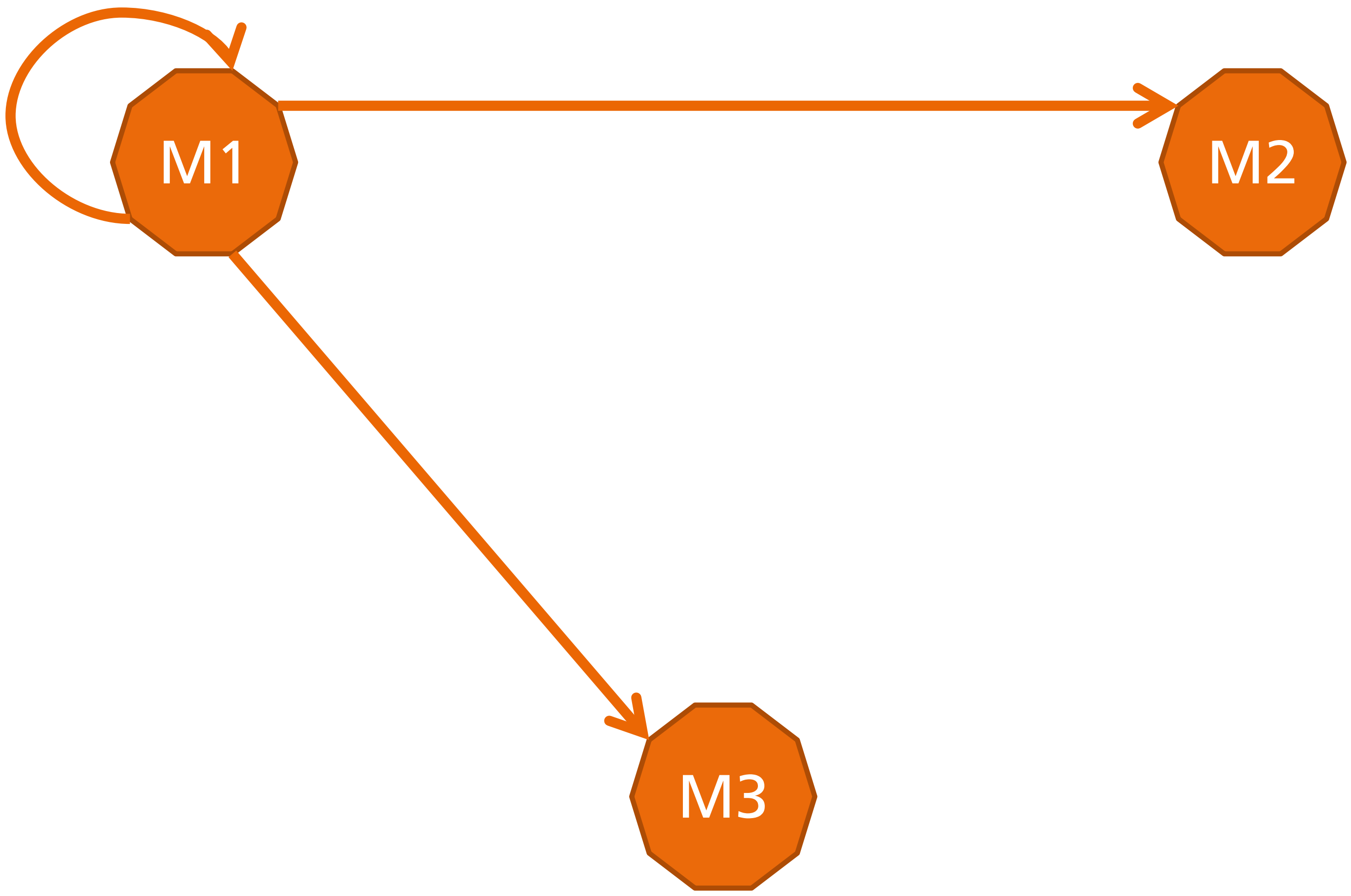

50 KM

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

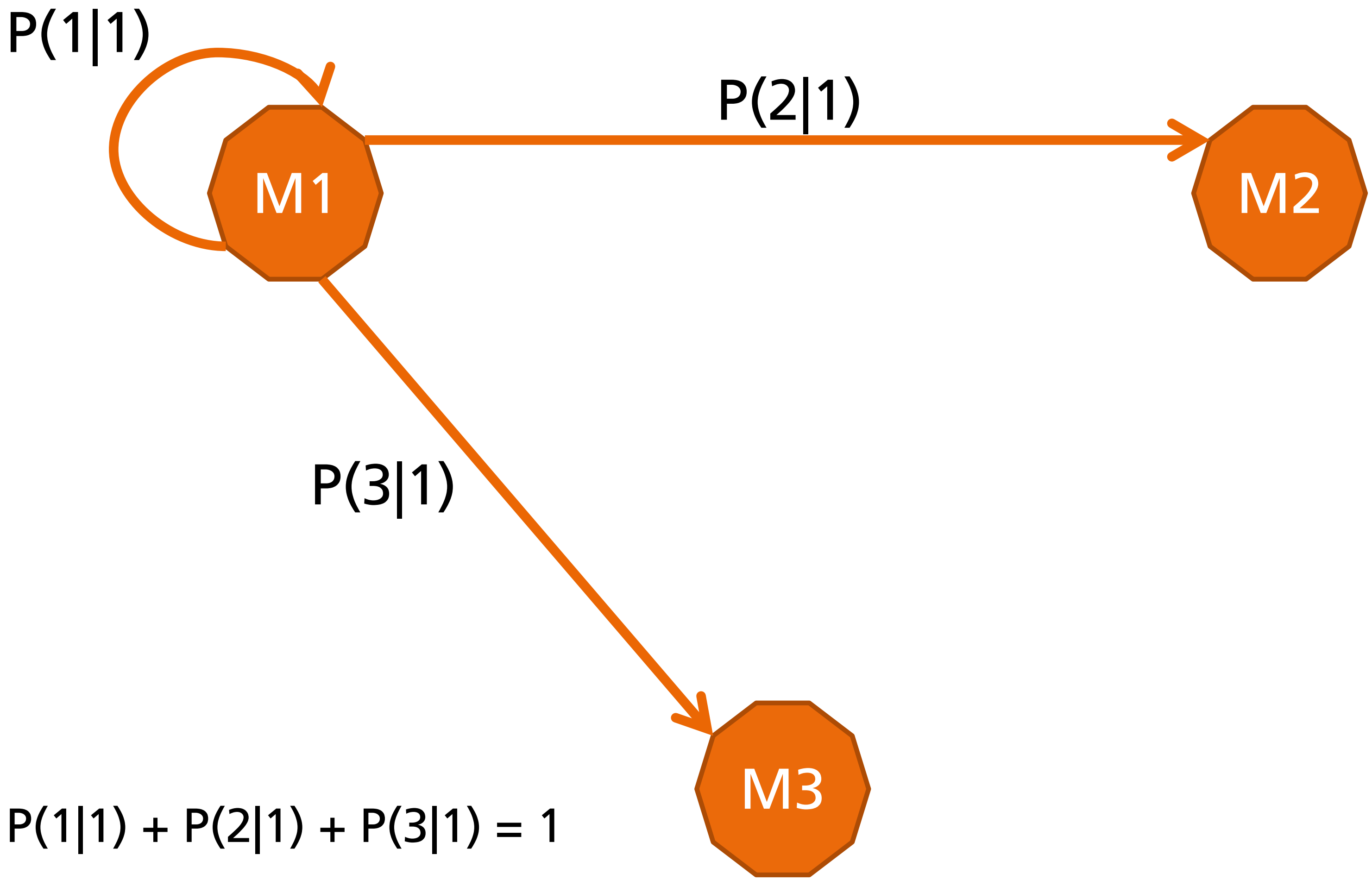
Quite general: agent switching between different modes of over-all behavior



Quite general: agent switching between different modes of over-all behavior



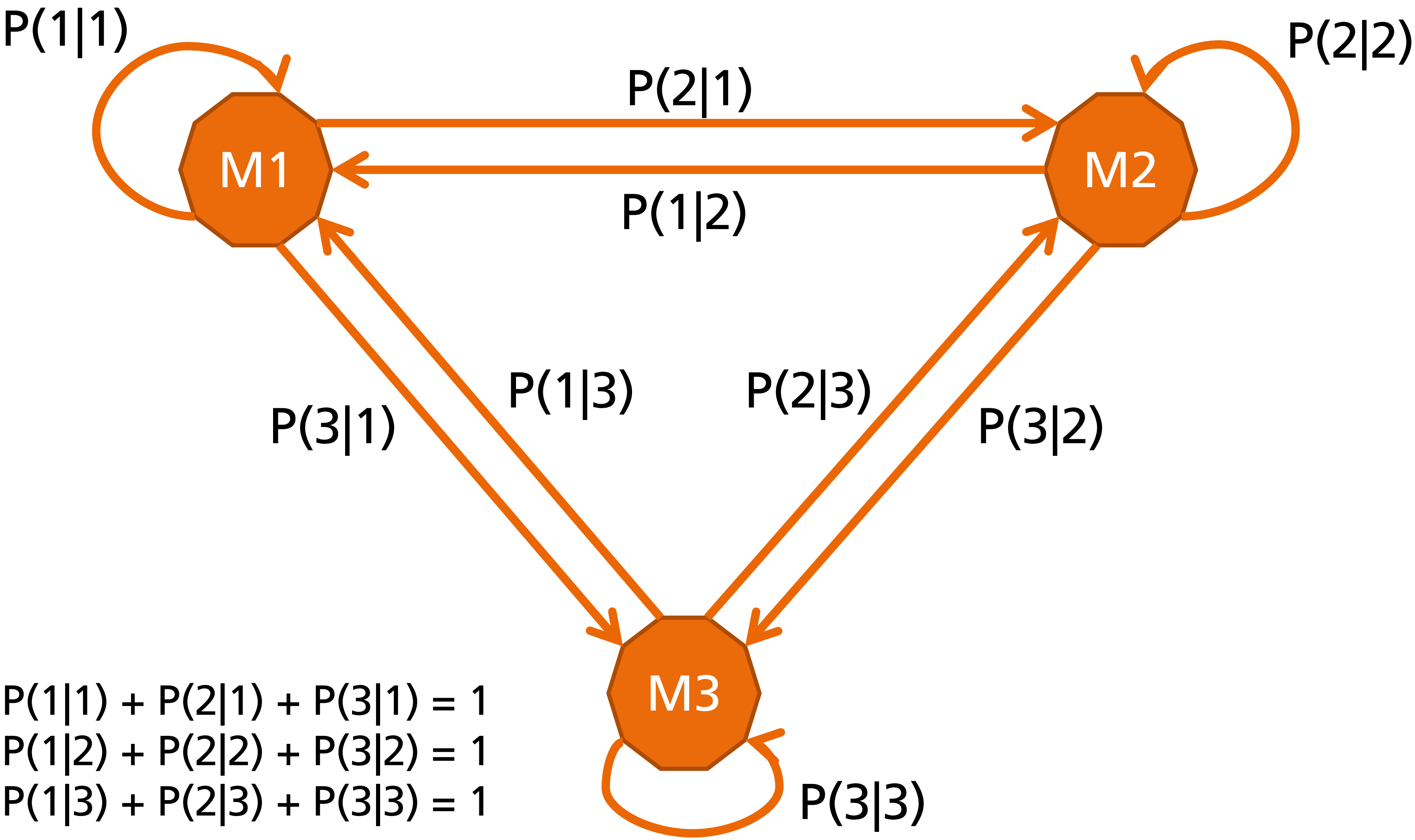
Quite general: agent switching between different modes of over-all behavior



P(1|1) + P(2|1) + P(3|1) = 1

Fraunhofer FKIE

Quite general: agent switching between different modes of over-all behavior



$$P(1|1) + P(2|1) + P(3|1) = 1$$
$$P(1|2) + P(2|2) + P(3|2) = 1$$
$$P(1|3) + P(2|3) + P(3|3) = 1$$

A quite general mathematical structure: a graph, characterized by nodes (here: evolution models) and directed edges defining an adjacency matrix (here: transition matrix P, stochastic matrix: columns sum up to one)

initial information on which model is currently being in effect: $\mathrm{p}_k = (p_k^1, p_k^2, p_k^3)^\top$

Markov propagation: $$\mathrm{p}_k = \mathrm{P}\,\mathrm{p}_{k-1} = \begin{pmatrix} p(1|1) & p(1|2) & p(1|3) \\ p(2|1) & p(2|2) & p(2|3) \\ p(3|1) & p(3|2) & p(3|3) \end{pmatrix} \begin{pmatrix} p_{k-1}^1 \\ p_{k-1}^2 \\ p_{k-1}^3 \end{pmatrix}$$

Perron-Frobenius: the spectral radius of stochastic matrices is 1, 1 is also an eigenvalue and the corresponding eigenvector is positive.

Exercise: Consider the example: $$\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

and calculate the invariant state (eigenvector for eingenvalue 1). Show numerically or mathematically that each initial state converges to the invariant state.

Excursus: Stochastic Characterization of Object Interrelations:

Estimation and Tracking of Adjacency Matrices

- Multiple object tracking: estimate from uncertain data Z at each time the kinematic state vector of all relevant objects: $p(x|Z)$.
- Sometimes of interest: interrelations between tracked objects. Example: reachability between two objects (communications, mutual help).
- Interrelations completely described by the adjacency matrix X of a graph (nodes: tracked objects, matrix elements: properties of the interrelation).
- Uncertainty of sensor data (kinematics, attributes) z, Z: adjacency matrix is a random matrix (matrix variate probability densities).
- State to be estimated: kinematics x of all objects, adjacency matrix X. Based on the sensor data, the knowledge on x, X is contained in: $p(x, X|z, Z)$.
- suitable families of matrix variate densities and likelihood functions: Bayes!

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1}) = p(\mathbf{x}_k|j_k, \mathbf{x}_{k-1}, j_{k-1})\, p(j_k|\mathbf{x}_{k-1}, j_{k-1}) \overset{!}{=} p(\mathbf{x}_k|\mathbf{x}_{k-1}, j_k)\, p(j_k|j_{k-1}) \quad \text{(Markov)}$$

$$= \underbrace{p(j_k|j_{k-1})}_{\text{interaction}}\ \underbrace{\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1}, j_k)\, p(j_k|j_{k-1}) \quad \text{(Markov)}$$

$$= \underbrace{p(j_k|j_{k-1})}_{\text{interaction}} \; \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

previous posterior written as a Gaussian mixture:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}^{j_{k-1}}_{k-1|k-1},\, \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),$$

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(x_k, j_k | x_{k-1}, j_{k-1}) = p(x_k | x_{k-1}, j_k)\, p(j_k | j_{k-1})$$
$$= \underbrace{p(j_k | j_{k-1})}_{\text{interaction}} \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

previous posterior written as a GAUSSian mixture:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}^{j_{k-1}}_{k-1|k-1},\, \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),$$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r} \sum_{j_{k-1}=1}^{r} \int d\mathbf{x}_{k-1}\, \underbrace{p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1})}_{\text{IMM dynamics}}\, p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1})$$

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(x_k, j_k | x_{k-1}, j_{k-1}) = p(x_k | x_{k-1}, j_k)\, p(j_k | j_{k-1})$$
$$= \underbrace{p(j_k | j_{k-1})}_{\text{interaction}} \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

previous posterior written as a Gaussian mixture:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}^{j_{k-1}}_{k-1|k-1},\, \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),$$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r}\sum_{j_{k-1}=1}^{r} \int d\mathbf{x}_{k-1}\, \underbrace{p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1})}_{\text{IMM dynamics}}\, p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1})$$
$$= \sum_{j_k=1}^{r}\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k j_{k-1}}_{k|k-1},\, \mathbf{P}^{j_k j_{k-1}}_{k|k-1}\big)$$

with $\mathbf{x}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}$, $\mathbf{P}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}\mathbf{F}^{j_k\top}_{k|k+1} + \mathbf{D}^{j_k}_{k|k+1}$ (product formula)

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(x_k, j_k|x_{k-1}, j_{k-1}) = p(x_k|x_{k-1}, j_k)\, p(j_k|j_{k-1})$$
$$= \underbrace{p(j_k|j_{k-1})}_{\text{interaction}} \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

previous posterior written as a GAUSSian mixture:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}^{j_{k-1}}_{k-1|k-1},\, \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),$$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r}\sum_{j_{k-1}=1}^{r} \int d\mathbf{x}_{k-1}\; \underbrace{p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1})}_{\text{IMM dynamics}}\; p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1})$$
$$= \sum_{j_k=1}^{r}\underbrace{\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k j_{k-1}}_{k|k-1},\, \mathbf{P}^{j_k j_{k-1}}_{k|k-1}\big)}_{\approx p(j_k|\mathcal{Z}^{k-1})\; \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k-1},\, \mathbf{P}^{j_k}_{k|k-1})}$$

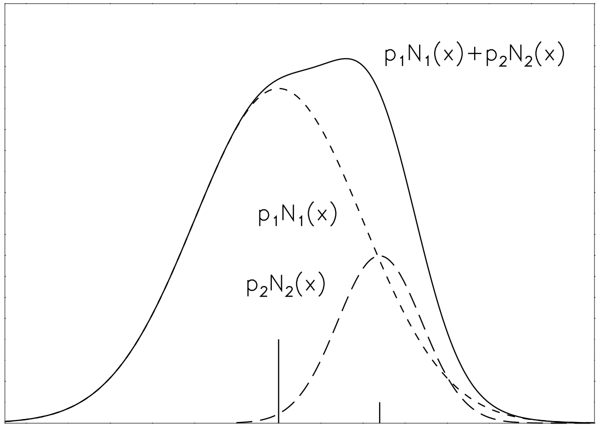
with $\mathbf{x}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}$, $\mathbf{P}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}\mathbf{F}^{j_k\top}_{k|k+1} + \mathbf{D}^{j_k}_{k|k+1}$ (product formula)

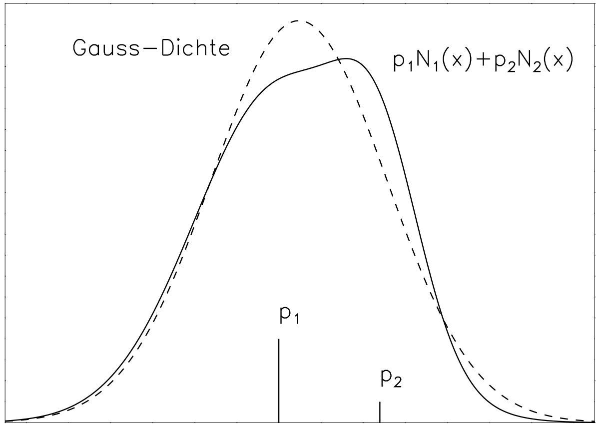
Approximate GAUSSian mixture representation of $p(\mathbf{x}_k|\mathcal{Z}^{k-1})$ with r mixture components!

Moment Matching: Approximate an arbitrary pdf

$p(x)$ with $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{C}[x] = \mathbf{P}$ by $p(x) \approx \mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{P}\big)$!

here especially: $p(x) = \sum_i p_i \mathcal{N}(x; \mathbf{x}_i, \mathbf{P}_i)$ (GAUSSian mixtures)





Exercise 5.1 Show:

$$\mathbf{x} = \sum_i p_i \, \mathbf{x}_i$$

$$\mathbf{P} = \sum_i p_i \left\{ \mathbf{P}_i + \overbrace{(\mathbf{x}_i - \mathbf{x})(\mathbf{x}_i - \mathbf{x})^\top}^{\text{spread term}} \right\}$$

BAYESian filtering update based on IMM predictions ($r = 1$: KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{\sum_{j_k=1}^{r} \ell(\mathbf{x}_k; Z_k, m_k)\, p(j_k|\mathcal{Z}^{k-1})\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1}^{j_k},\ \mathbf{P}_{k|k-1}^{j_k}\big)}{\sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k-1})\ \int d\mathbf{x}_k\ \ell(\mathbf{x}_k; Z_k, m_k)\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1}^{j_k},\ \mathbf{P}_{k|k-1}^{j_k}\big)}$$

Fraunhofer FKIE

Bayesian filtering update based on IMM predictions ($r = 1$: KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{\sum_{j_k=1}^{r} \ell(\mathbf{x}_k; Z_k, m_k)\, p(j_k|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k}\big)}{\sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k-1})\, \int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k}\big)}$$

Consider as a simple example $\ell(\mathbf{x}_k; Z_k, m_k) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$!

$$= \sum_{j_k=1}^{r} \frac{p(j_k|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k})}{\sum_{j'_k=1}^{r} p(j'_k|\mathcal{Z}^{k-1})\, \int d\mathbf{x}_k\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j'_k},\, \mathbf{P}_{k|k-1}^{j'_k}\big)}$$

$$= \sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^k)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k}^{j_k},\, \mathbf{P}_{k|k}^{j_k}\big) \qquad \text{(due to the product formula)}$$

Bayesian filtering update based on IMM predictions ($r = 1$: KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k;\, Z_k, m_k)\; p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\; \ell(\mathbf{x}_k;\, Z_k, m_k)\; p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{\sum_{j_k=1}^{r} \ell(\mathbf{x}_k;\, Z_k, m_k)\; p(j_k|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k-1},\, \mathbf{P}^{j_k}_{k|k-1}\big)}{\sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k-1})\; \int d\mathbf{x}_k\; \ell(\mathbf{x}_k;\, Z_k, m_k)\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k-1},\, \mathbf{P}^{j_k}_{k|k-1}\big)}$$

Consider as a simple example $\ell(\mathbf{x}_k;\, Z_k, m_k) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$!

$$= \sum_{j_k=1}^{r} \frac{p(j_k|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\; \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k-1},\, \mathbf{P}^{j_k}_{k|k-1})}{\sum_{j'_k=1}^{r} p(j'_k|\mathcal{Z}^{k-1})\; \int d\mathbf{x}_k\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j'_k}_{k|k-1},\, \mathbf{P}^{j'_k}_{k|k-1}\big)}$$

$$= \sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^k)\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k},\, \mathbf{P}^{j_k}_{k|k}\big) \qquad \text{(due to the product formula)}$$

with: $$p(j_k|\mathcal{Z}^k) = \frac{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1}, \mathbf{H}_k\mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k\big)}{\sum_{j'_k=1}^{r} p(j'_k|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}^{j'_k}_{k|k-1}, \mathbf{H}_k\mathbf{P}^{j'_k}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k\big)} \qquad \text{(mixture coefficients)}$$

$$\mathbf{x}^{j_k}_{k|k} = \mathbf{x}^{j_k}_{k|k-1} + \mathbf{W}^{j_k}_{k|k}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1}), \qquad \mathbf{W}^{j_k}_{k|k} = \mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k^{\top}\mathbf{S}^{j_k^{-1}}_{k|k} \qquad \text{(KALMAN update)}$$

$$\mathbf{P}^{j_k}_{k|k} = \mathbf{P}^{j_k}_{k|k-1} - \mathbf{W}^{j_k}_{k|k-1}\mathbf{S}^{j_k}_{k|k}\mathbf{W}^{j_k}_{k|k-1}, \qquad \mathbf{S}^{j_k}_{k|k} = \mathbf{H}_k\mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k.$$

multiple models: $p(x_k, j_k | x_{k-1}, j_{k-1}) = p(j_k|j_{k-1})\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}^{j_k}_{k|k-1}\big)$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r}\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}^{j_k j_{k-1}}_{k|k-1},\ \mathbf{P}^{j_k j_{k-1}}_{k|k-1}\big)$$

$$\mathbf{x}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}, \qquad \mathbf{P}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}\mathbf{F}^{j_k\top}_{k|k+1} + \mathbf{D}^{j_k}_{k|k+1}$$

mixing step: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k-1})\ \mathcal{N}(\mathbf{x}_k;\ \mathbf{x}^{j_k}_{k|k-1},\ \mathbf{P}^{j_k}_{k|k-1})$$

$$p(j_k|\mathcal{Z}^{k-1}) = \textstyle\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})$$

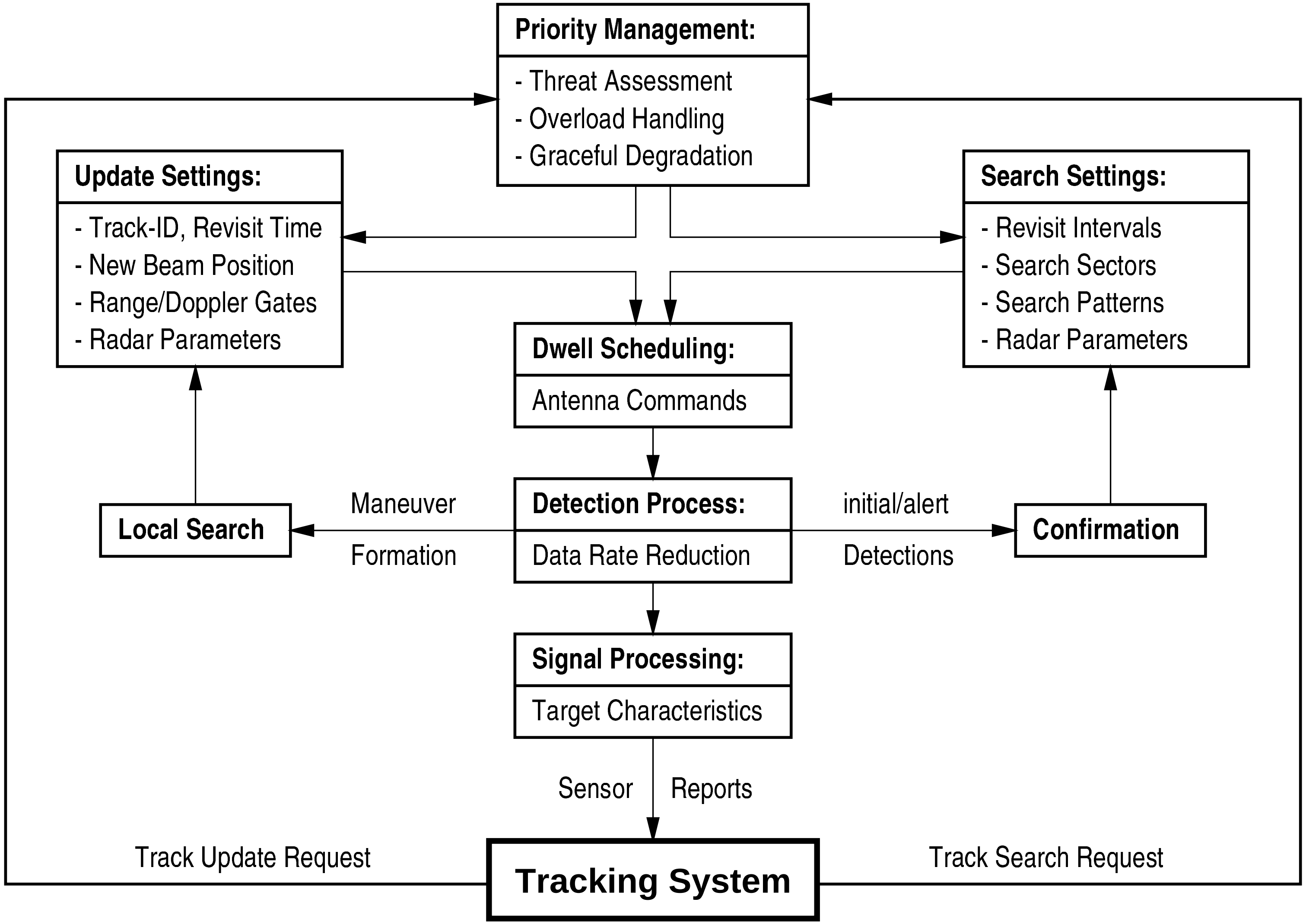
$$\mathbf{x}^{j_k}_{k|k-1} = \frac{1}{p(j_k|\mathcal{Z}^{k-1})}\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})\mathbf{x}^{j_k j_{k-1}}_{k|k-1}, \qquad \mathbf{P}^{j_k}_{k|k-1} = \frac{1}{p(j_k|\mathcal{Z}^{k-1})}\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})\Big(\mathbf{P}^{j_k j_{k-1}}_{k|k-1} + (\mathbf{x}^{j_k j_{k-1}}_{k|k-1} - \mathbf{x}^{j_k}_{k|k-1})(\cdots)^\top\Big)$$

filtering: $$p(\mathbf{x}_k|\mathcal{Z}^{k}) = \sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k})\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}^{j_k}_{k|k},\ \mathbf{P}^{j_k}_{k|k}\big)$$

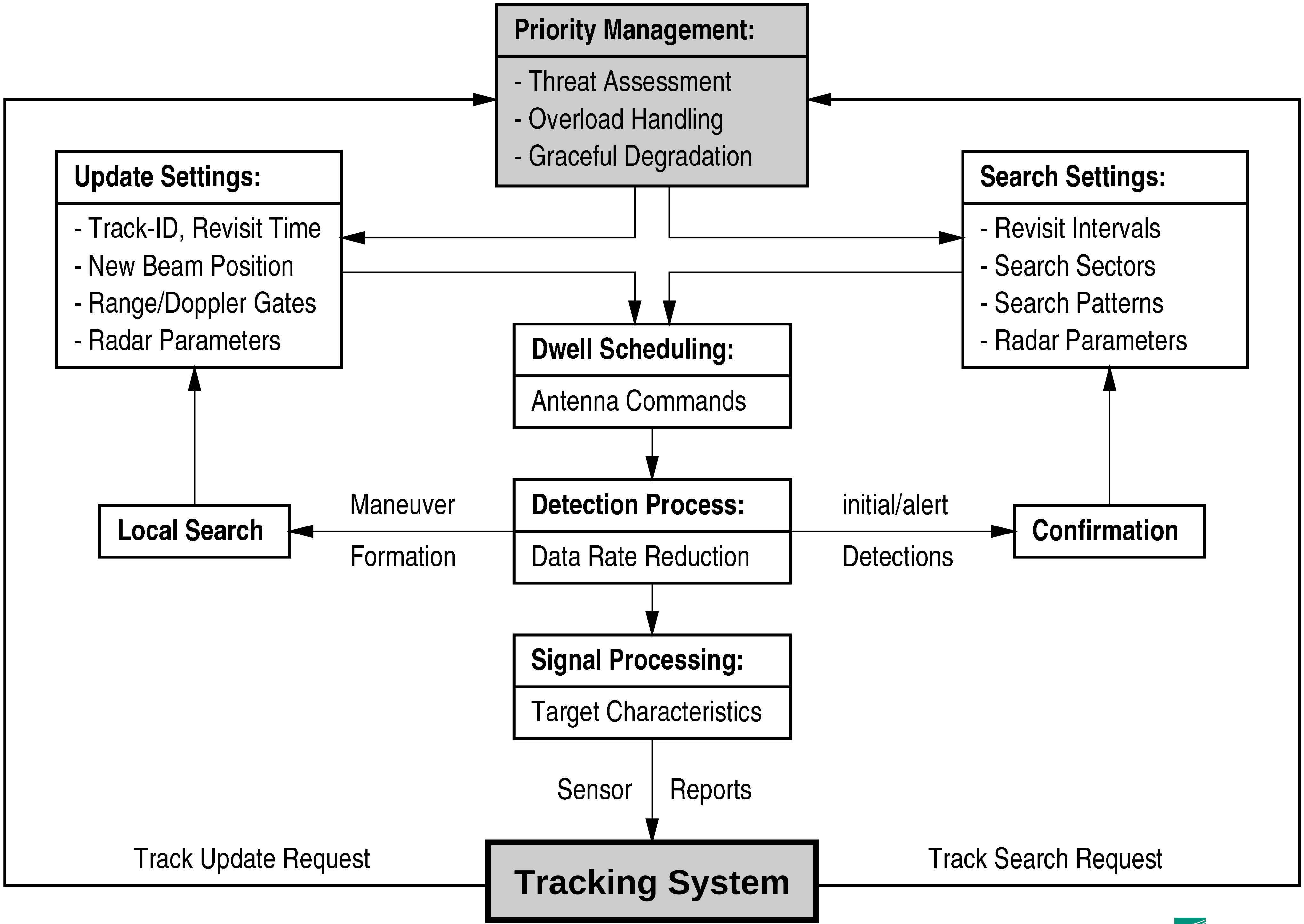
with: $$p(j_k|\mathcal{Z}^k) = \frac{\mathcal{N}\Big(\mathbf{z}_k;\ \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1},\ \mathbf{H}_k\mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k\Big)}{\sum_{j'_k=1}^{r} p(j'_k|\mathcal{Z}^{k-1})\ \mathcal{N}\Big(\mathbf{z}_k;\ \mathbf{H}_k\mathbf{x}^{j'_k}_{k|k-1},\ \mathbf{H}_k\mathbf{P}^{j'_k}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k\Big)}$$ (mixture coefficients)

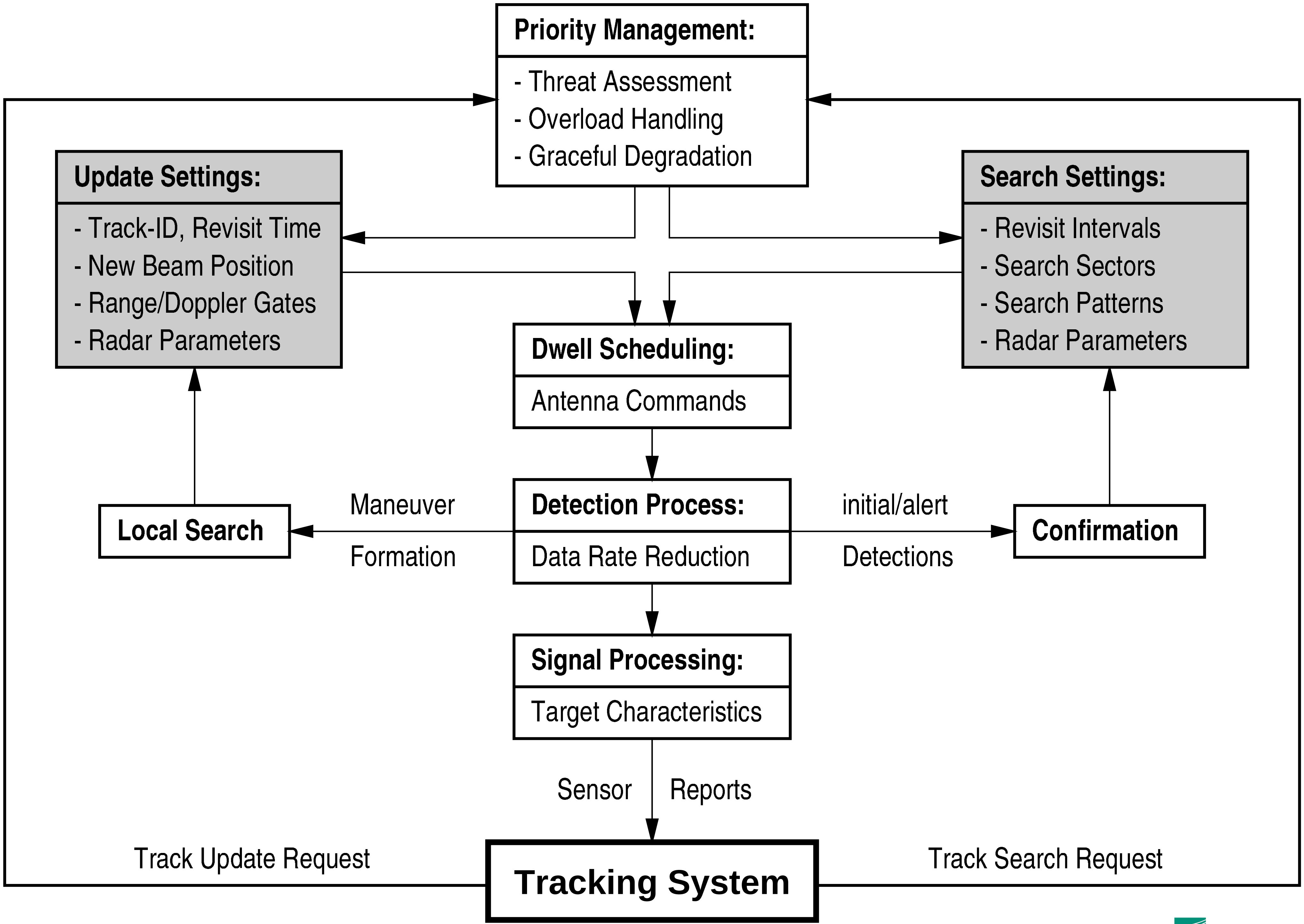
$$\mathbf{x}^{j_k}_{k|k} = \mathbf{x}^{j_k}_{k|k-1} + \mathbf{W}^{j_k}_{k|k}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}^{j_k}_{k|k-1}), \quad \mathbf{P}^{j_k}_{k|k} = \mathbf{P}^{j_k}_{k|k-1} - \mathbf{W}^{j_k}_{k|k-1}\mathbf{S}^{j_k}_{k|k}\mathbf{W}^{j_k}_{k|k-1}, \quad \mathbf{W}^{j_k}_{k|k} = \mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}^{j_k^{-1}}_{k|k} \quad \mathbf{S}^{j_k}_{k|k} = \mathbf{H}_k\mathbf{P}^{j_k}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k$$

Radar Function Control: Information Flow



Fraunhofer FKIE


Priority Management:
- Threat Assessment
- Overload Handling
- Graceful Degradation
Update Settings:
- Track-ID, Revisit Time
- New Beam Position
- Range/Doppler Gates
- Radar Parameters
Search Settings:
- Revisit Intervals
- Search Sectors
- Search Patterns
- Radar Parameters
Dwell Scheduling:
Antenna Commands
Local Search
Maneuver
Formation
Detection Process:
Data Rate Reduction
initial/alert
Detections
Confirmation
Signal Processing:
Target Characteristics
Sensor
Reports
Track Update Request
Tracking System
Track Search Request


Priority Management:
- Threat Assessment
- Overload Handling
- Graceful Degradation
Update Settings:
- Track-ID, Revisit Time
- New Beam Position
- Range/Doppler Gates
- Radar Parameters
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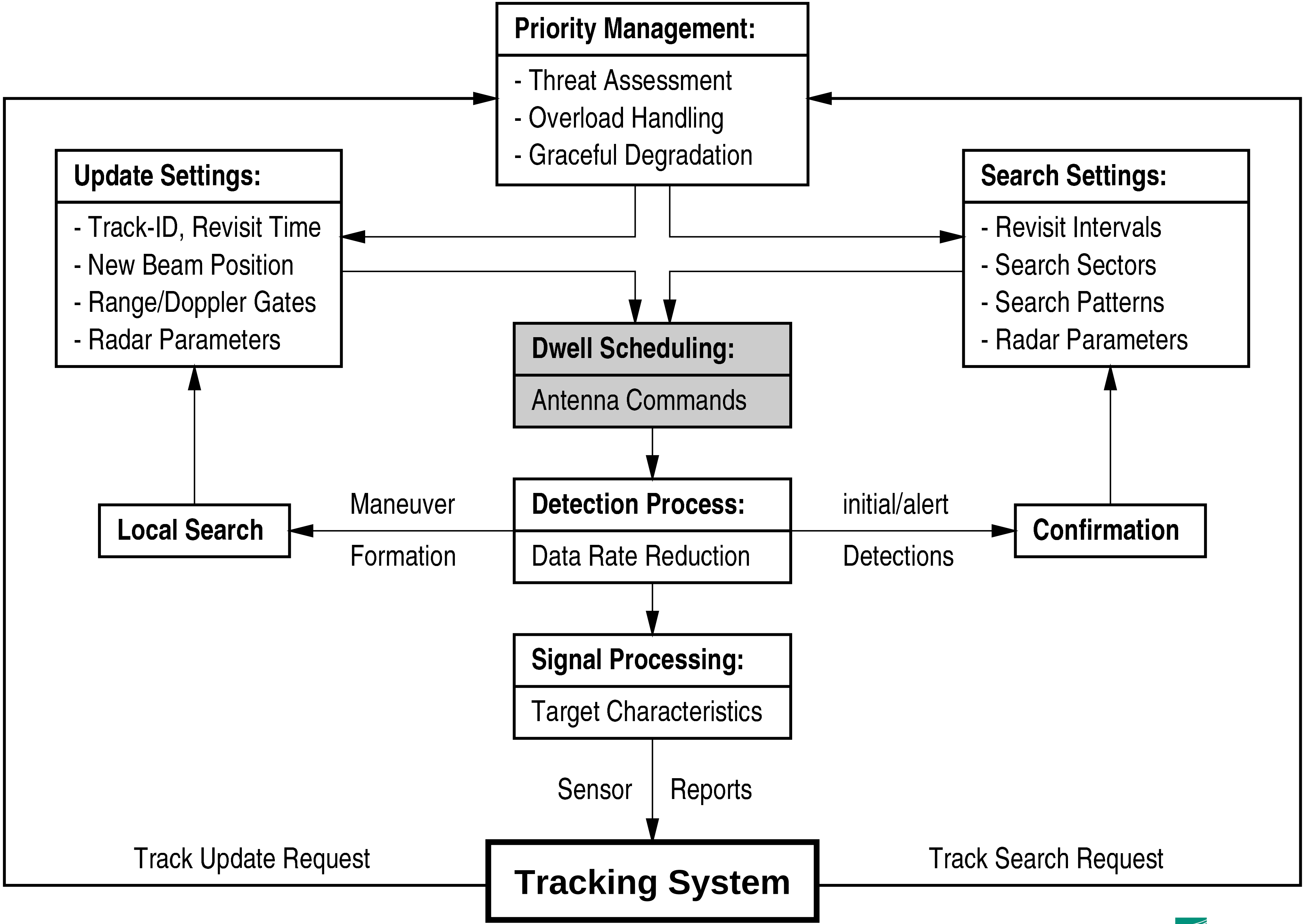
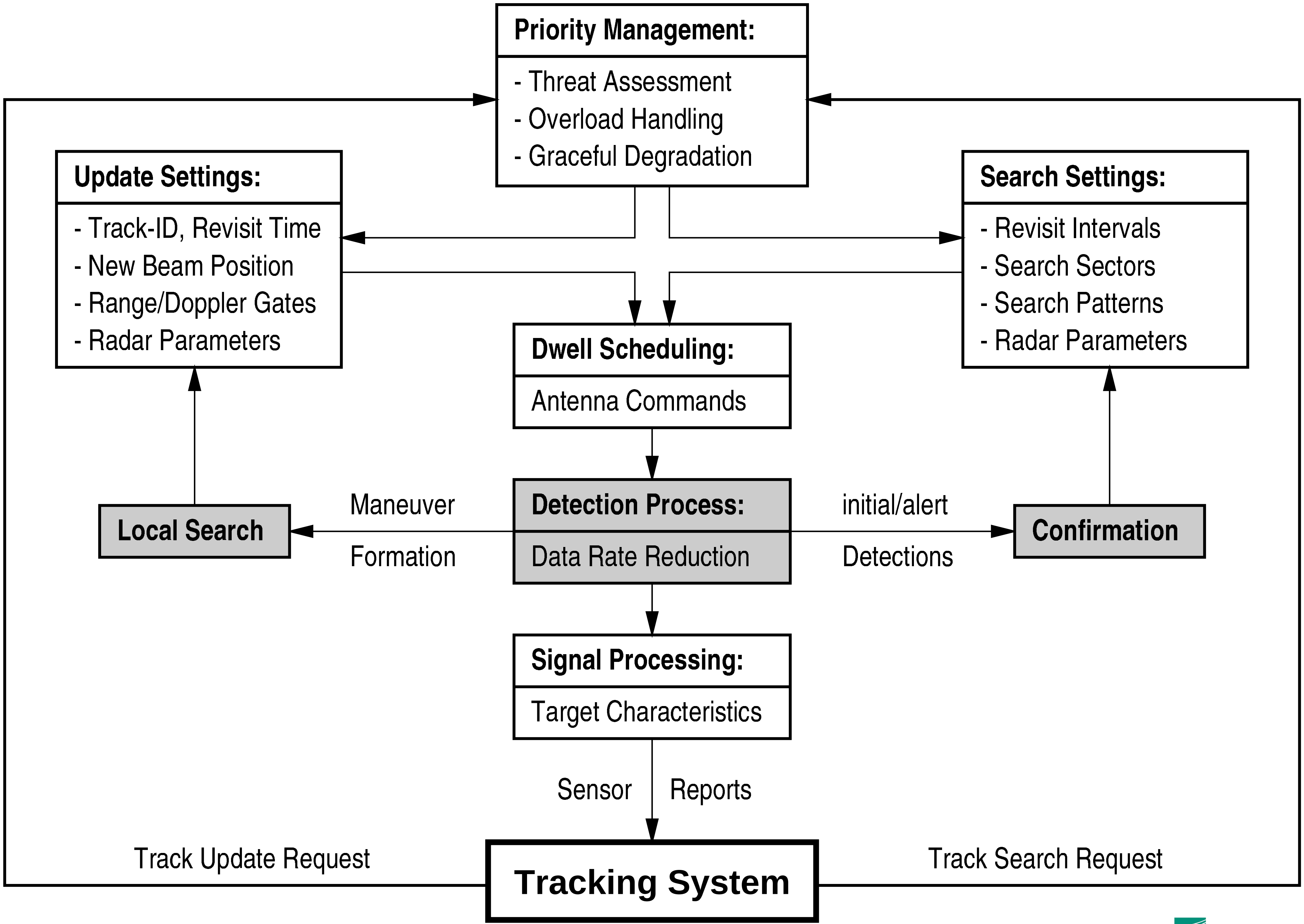


Fraunhofer FKIE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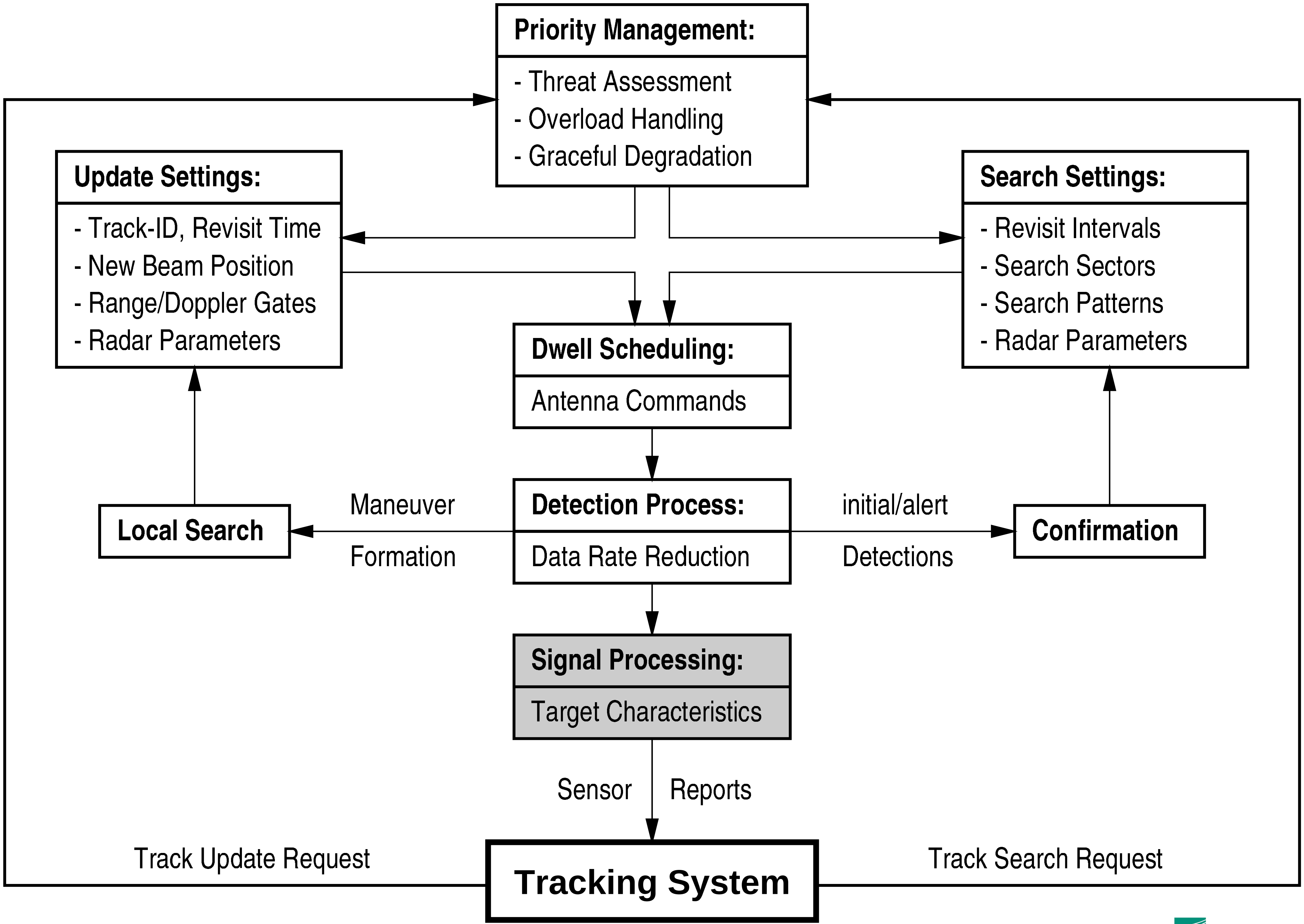
Priority Management:
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Reports
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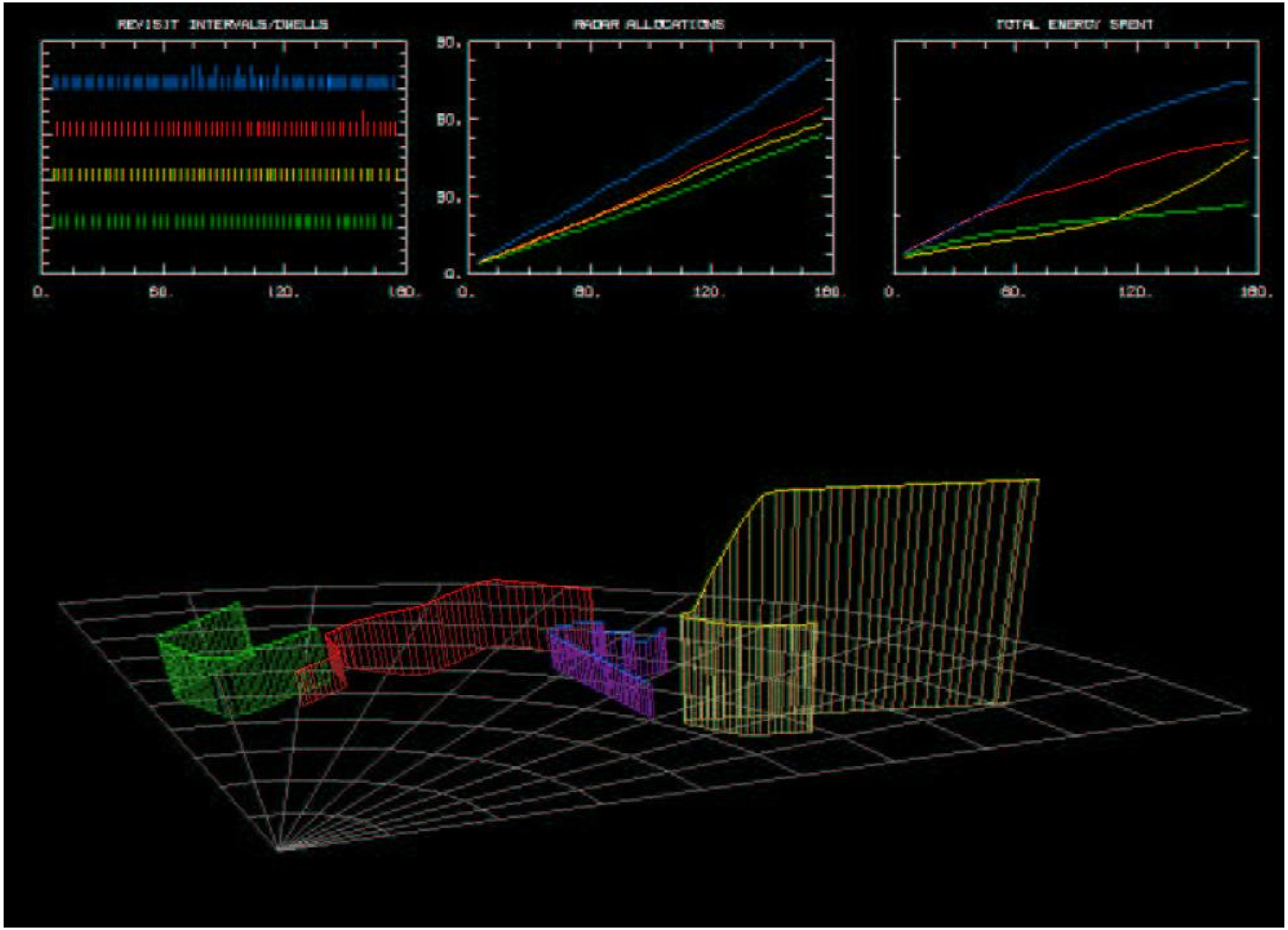
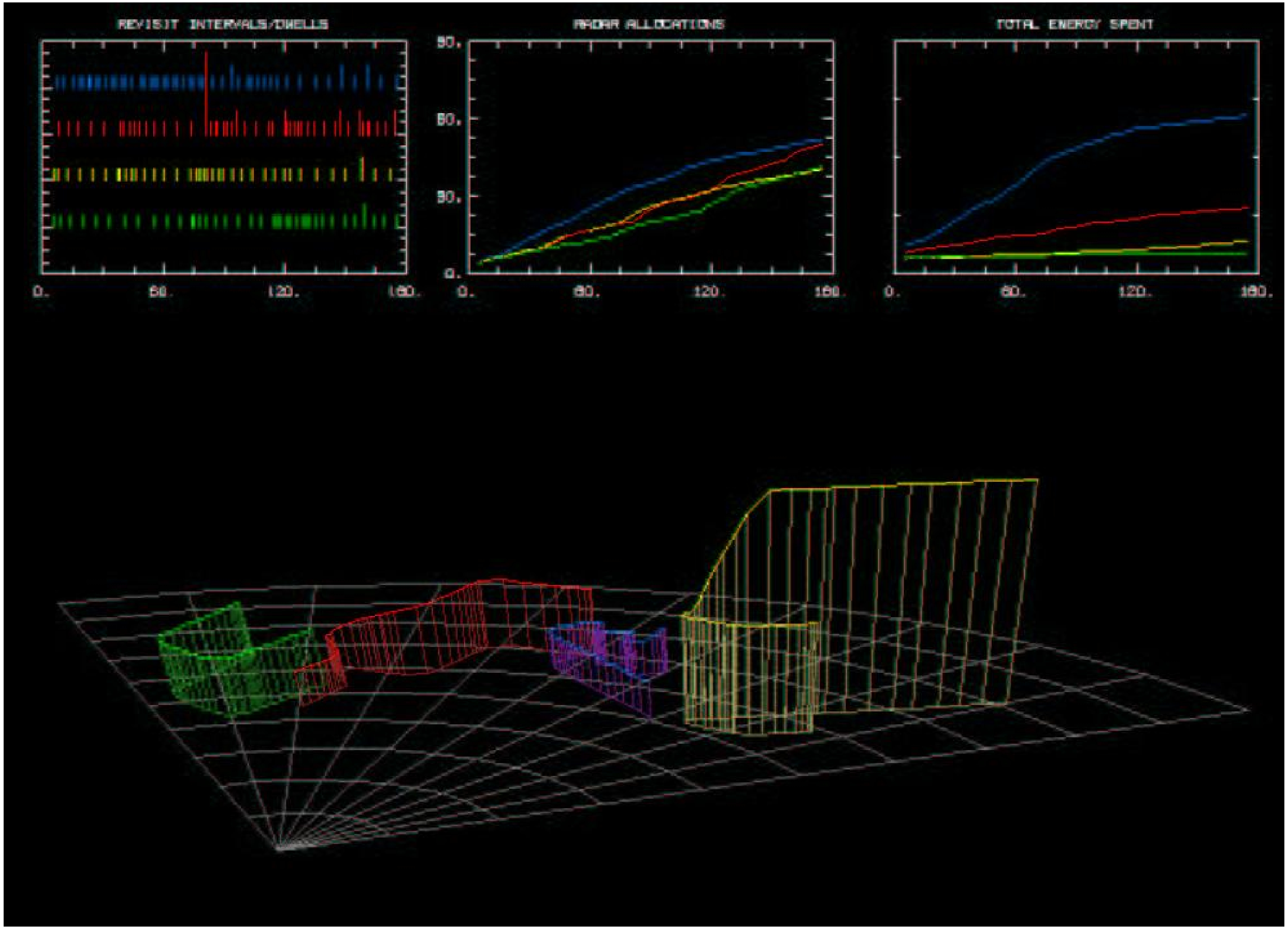
Tracking System
Track Search Request

Phased-Array Tracking: Standard Sensor Management



Phased-Array Tracking: Adaptive Sensor Management



Fraunhofer FKIE

IMM Models: Retrodiction

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k) \qquad \text{(marginal density!)}$$

IMM Models: Retrodiction

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k) \qquad \text{(marginal density!)}$$

$$= \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}\mathcal{Z}^k)\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction for } t_{l+1}}$$

for time $l+1$ assume: $p(\mathbf{x}_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1}}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}^k_{i_{l+1}}, \mathbf{P}^k_{i_{l+1}}\big)$

IMM Models: Retrodiction

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k) \qquad \text{(marginal density!)}$$

$$= \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}\mathcal{Z}^k)\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{Retrodiction for } t_{l+1}}$$

for time $l+1$ assume: $p(\mathbf{x}_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1}}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}^k_{i_{l+1}},\, \mathbf{P}^k_{i_{l+1}}\big)$

$$p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^k}) = p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^l}) = \frac{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)\; p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}{\sum_{i_l} \int \mathbf{x}_l\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)}_{\text{IMM model}}\, \underbrace{p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}_{\text{filtering } t_l}}$$

IMM Models: Retrodiction

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1},i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k) \qquad \text{(marginal density!)}$$

$$= \sum_{i_{l+1},i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}\mathcal{Z}^k)\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction for } t_{l+1}}$$

for time $l+1$ assume: $p(\mathbf{x}_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1}}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}^k_{i_{l+1}},\, \mathbf{P}^k_{i_{l+1}}\big)$

$$p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^k}) = p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^l}) = \frac{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)\, p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}{\sum_{i_l}\int \mathbf{x}_l\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)}_{\text{IMM model}}\, \underbrace{p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}_{\text{filtering } t_l}}$$

$$= \mathbf{c}_{i_{l+1}i_l}(\mathbf{x}_{l+1})\, \mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{i_l} + \mathbf{W}_{i_{l+1}i_l}(\mathbf{x}_{l+1} - \mathbf{x}_{i_{l+1}i_l}),\, \mathbf{P}_{i_l} - \mathbf{W}_{i_{l+1}i_l}\mathbf{P}_{i_{l+1}i_l}\mathbf{W}_{i_{l+1}i_l}{}^\top\big) \qquad \textbf{product formula!}$$

with: $\mathbf{c}_{i_{l+1}i_l}(\mathbf{x}_{l+1}) = \dfrac{\boldsymbol{\mu}_{i_{l+1}i_l}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{i_{l+1}i_l},\, \mathbf{P}_{i_{l+1}i_l}\big)}{\sum_{i_l}\boldsymbol{\mu}_{i_{l+1}i_l}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{i_{l+1}i_l},\, \mathbf{P}_{i_{l+1}i_l}\big)}$ $\qquad \mathbf{W}_{i_{l+1}i_l} = \mathbf{P}_{i_l}\mathbf{F}^\top_{i_{l+1}}(\mathbf{F}_{i_{l+1}}\mathbf{P}_{i_l}\mathbf{F}^\top_{i_{l+1}} + \mathbf{D}_{i_{l+1}})^{-1}$

$$\approx \mathbf{c}_{i_{l+1}i_l}(\mathbf{x}^k_{i_{l+1}})$$

Fraunhofer FKIE

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

$$= \sum_{i_{l+1}, i_l} \boldsymbol{\mu}^k_{i_{l+1} i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\ \mathbf{x}^k_{i_{l+1} i_l},\ \mathbf{P}^k_{i_{l+1} i_l}\Big)$$

exponential growth of dynamics histories $i_{i_{l+1} i_l \dots}$!

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

$$= \sum_{i_{l+1}, i_l} \boldsymbol{\mu}^k_{i_{l+1} i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_{l+1} i_l},\, \mathbf{P}^k_{i_{l+1} i_l}\Big)$$

exponential growth of dynamics histories $i_{i_{l+1} i_l \ldots}$!

$$= \sum_{i_l} \underbrace{\sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1} i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_{l+1} i_l},\, \mathbf{P}^k_{i_{l+1} i_l}\Big)}_{\text{approximation: moment matching!}}$$

finally: $p(\mathbf{x}_l|\mathcal{Z}^k) \approx \sum_{i_l} \boldsymbol{\mu}^k_{i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_l},\, \mathbf{P}^k_{i_l}\Big)$

generalize: model histories of *variable* length!

IMM Modeling: Suboptimal Realization

- **Conventional KALMAN filtering**

  Only *one* component: worst-case assumtion

- **standard IMM filter (as discussed!)**

  Approximate after prediction, *before* update by r components! Effort: $\sim r$ KALMAN filter

- **GPB: Generalized Pseudo-BAYESian**

  Approximate *after* measurement processing by r components! Effort: $\sim r^2$ KALMAN filter

- **IMM-MHT filter (nearly optimal)**

  Accept longer dynamics histories $\rightarrow$ *variable* number of components!

Extendable to ambiguity with respect to sensor models!