Recapitulation: Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'KALMAN gain matrix'}$$

Exercise 3.6

In your sensor simulator, chose a sensor at position $\mathbf{r}_s$, for example $\mathbf{r}_s = (0, 0)^\top$, that produces measurements $\mathbf{z}_k$ of the Cartesian target positions $\mathbf{H}\mathbf{x}_k$ from your ground truth generator. Use the measurement covariance matrix $\mathbf{R} = \sigma_c^2 \mathtt{diag}[1, 1]$, σ_c = 50 m, for all measurements, but allow individual measurement error covariances for each measurement. Program your first Kalman filter using a constant acceleration. Visualize your results nicely! Compare the ground truth, the measurement, and the estimates!

Fraunhofer FKIE

a first remark on initiation: $p(\mathbf{x}_0|\mathcal{Z}^0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, $\mathbf{P}_{0|0}$ 'large'

$$\mathbf{x}_{0|0} = \begin{pmatrix} \mathbf{r}_{0|0} \\ \dot{\mathbf{r}}_{0|0} \\ \ddot{\mathbf{r}}_{0|0} \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{P}_{0|0} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (v_{max})^2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (q_{max})^2\mathbf{1} \end{pmatrix}$$

position information: first measurement $\mathbf{z}_0$, ignorance = measurement error $\mathbf{R}$!

ignorance on velocity: sphere with radius v_{max} around zero
(= no information on direction, but on 'limits')

ignorance on acceleration: sphere with radius q_{max} around zero

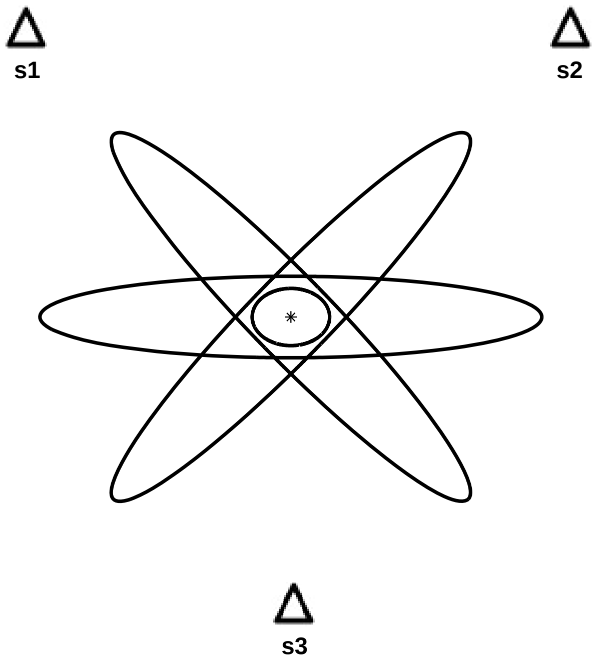
Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

a closer look: The error of each measurement $\mathbf{z}_i$ is described by a related measurement error *covariance matrix* $\mathbf{R}_i$ ('error ellipsoids'). In 2 dimensions:



$\mathbf{R}_i$ can strongly depend on the underlying senor-to-target geometry!

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

  $\mathbf{z}_k = (r_k, \varphi_k)^\top$, measurement error: $\mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}$, r, φ independent

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \text{ measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

- **What is the likelihood function in Cartesian coordinates?**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix}$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

constant and linear term of a Taylor series only, blackboard!

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \quad r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

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- **affine transform of GAUSSian random variables:**

$$\mathcal{N}\big(\mathbf{z};\ \mathbf{x},\ \mathbf{R}\big) \xrightarrow{\mathbf{z}'=\mathbf{t}+\mathbf{T}\mathbf{z}} \mathcal{N}\big(\mathbf{z}';\ \mathbf{t}+\mathbf{T}\mathbf{x},\ \mathbf{T}\mathbf{R}\mathbf{T}^\top\big)$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

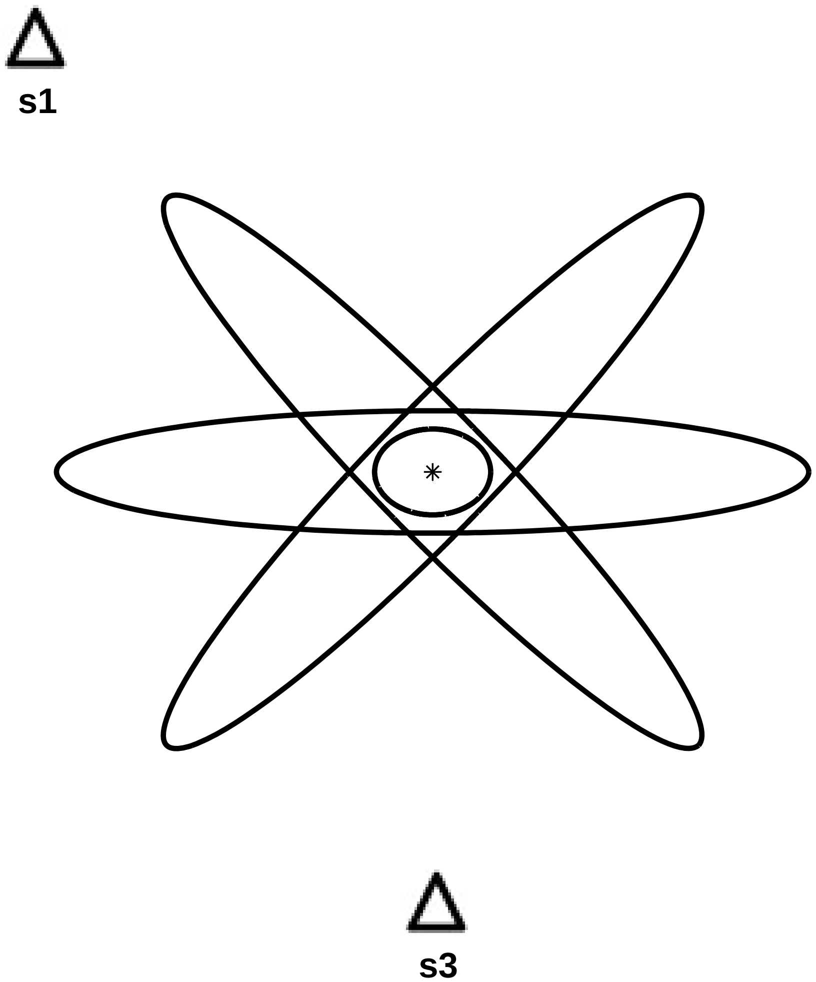
$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **Cartesian error covariance (time dependent):**

$$\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\, \mathbf{R}\, \mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top$$

- **sensor fusion: sensor-to-target-geometry enters into $\mathbf{T}\mathbf{R}\mathbf{T}^\top$**



sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathsf{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathsf{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$\begin{aligned} p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) &= p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \qquad \text{independent sensors} \\ &= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big) \end{aligned}$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathsf{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$p(\mathbf{z}_k^1, \mathbf{z}_k^2|\mathbf{x}_k) = p(\mathbf{z}_k^1|\mathbf{x}_k)\, p(\mathbf{z}_k^2|\mathbf{x}_k) \qquad \text{independent sensors}$$

$$= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big)$$

Fraunhofer FKIE

A Useful Product Formula for GAUSSians

$$\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\, \mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Fraunhofer FKIE

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$\begin{aligned}
p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) &= p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \qquad \text{independent sensors} \\
&= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big) \\
&= \mathcal{N}\big(\mathbf{H}_k \mathbf{x}_k;\, \mathbf{z}_k^1,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big) \\
&\propto \mathcal{N}\Big(\mathbf{H}_k \mathbf{x}_k;\, \underbrace{\mathbf{R}_k(\mathbf{R}_k^{1\,-1}\mathbf{z}_k^1 + \mathbf{R}_k^{2\,-1}\mathbf{z}_k^2)}_{=\mathbf{z}_k},\, \underbrace{(\mathbf{R}_k^{1\,-1} + \mathbf{R}_k^{2\,-1})^{-1}}_{=\mathbf{R}_k}\Big) \\
&= \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k\big)
\end{aligned}$$

Exercise 4.1 Generalize to the case $S_k > 2$ (induction argument)!

One possible fusion strategy: Create a singe *effective measurement* by preprocessing of the individual measurements!

$$\mathbf{z}_k = \mathbf{R}_k \sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1} \mathbf{z}_k^s \qquad \text{weighted arithmetic mean of measurements}$$

$$\mathbf{R}_k = \left(\sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1}\right)^{-1} \qquad \text{harmonic mean of measuremt covariances}$$

A typical structure for fusion equations!

With measurement specific measurement error covariances, your Kalman filter already is a multiple sensor fusion algorithms. Try and play!

Recapitulation: A popular model for object evolutions

Piecewise Constant White Acceleration Model

$$p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\Big(\mathbf{x}_k;\ \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}_{k|k-1}\Big)$$

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

Consider state vectors $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$ (position, velocity, acceleration)

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\mathbf{I} & \Delta T_k\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^2\,\mathbf{I} & \Delta T_k\mathbf{I} & \mathbf{I} \end{pmatrix}$$

with $\Delta T_k = t_k - t_{k-1}$. Reasonable choice: $\frac{1}{2}v_{\text{max}}/v_{\text{max}} \le \Sigma_k \le v_{\text{max}}/q_{\text{max}}$

Another, rather realistic model (van Keuk):

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} & \frac{1}{2}(t_k - t_{k-1})^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-(t_k - t_{k-1})/\theta}\,\mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \mathrm{diag}[1,1,1]$$

$$\mathbf{D}_{k|k-1} = \Sigma^2(1 - \mathrm{e}^{-2(t_k - t_{k-1})/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{O} = \mathrm{diag}[0,0,0]$$

maneuver correlation time θ (z.B. 60 s), *limiting acceleration* Σ (z.B. 2 g)

There are many different evolution models adapted to particular problems!

Exercise 4.2 (voluntary!)

Show for the acceleration process:

$$\mathbb{E}[\ddot{\mathbf{r}}_k] = 0, \quad \mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top] = \Sigma^2\,\mathrm{e}^{-(t_k - t_l)/\theta}\,\mathbf{I}, \; l \leq k$$

$\mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top]$ is called ‘auto correlation function’.

Recapitulation: measurement process

- **linear measurement equation:**

  $$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{u}_k, \quad p(\mathbf{u}_k) = \mathcal{N}\big(\mathbf{u}_k;\ \mathbf{o},\ \mathbf{R}_k\big)$$

  - to be measured: *linear* functions of the object state
  - measurement error: biasfree, Gaussian distrib.
    independent for different t_k
  - $\mathbf{y}_k = \mathbf{z}_k - \mathbf{H}_k \mathbf{x}_k$ has the pdf: $p(\mathbf{y}_k) = p(\mathbf{u}_k)$

- **Approach for the requested pdf ('likelihood fkt.):**

  $$\boxed{p\big(\mathbf{z}_k | \, \mathbf{x}_k\big) = \mathcal{N}\big(\mathbf{z}_k;\ \mathbf{H}_k \mathbf{x}_k,\ \mathbf{R}_k\big)}$$

- **Example: position measurement**

  $$\mathbf{H}_k = (\mathbf{I}, \mathbf{O}, \mathbf{O}), \quad \mathbf{H}_k \mathbf{x}_k = \mathbf{r}_k$$

  $\mathbf{R}_k$: measurement error covariance matrix

  possibly depending on the sensor-to-target geometry

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p(\mathbf{x}_l|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k)\; \underbrace{p(\mathbf{x}_{l+1}|\,\mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

Fraunhofer FKIE

Retrodiction: How to calculate the pdf $p(\mathrm{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathrm{x}_l|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l, \mathrm{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big)\ \underbrace{p\big(\mathrm{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

$$p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big) = \frac{p(Z_k, \dots, Z_{l+1}|\mathrm{x}_{l+1}, \mathrm{x}_l, \mathcal{Z}^l)\; p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)}{\int d\mathrm{x}_l\; p(Z_k, \dots, Z_{l+1}|\mathrm{x}_{l+1}, \mathrm{x}_l, \mathcal{Z}^l)\; p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)} = p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p(\mathbf{x}_l|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k)\ \underbrace{p(\mathbf{x}_{l+1}|\,\mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

$$p(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^k) = p(\mathbf{x}_l|\,\mathbf{x}_{l+1}, \mathcal{Z}^l) = \frac{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)\ p(\mathbf{x}_l|\,\mathcal{Z}^l)}{\int d\mathbf{x}_l\ \underbrace{p(\mathbf{x}_{l+1}|\,\mathbf{x}_l)}_{\text{dynamics model}}\ \underbrace{p(\mathbf{x}_l|\,\mathcal{Z}^l)}_{\text{filtering } t_l}}$$

Retrodiction: How to calculate the pdf $p(\mathrm{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p(\mathrm{x}_l|\,\mathcal{Z}^k) = \int d\mathrm{x}_{l+1}\, p(\mathrm{x}_l, \mathrm{x}_{l+1}|\,\mathcal{Z}^k) = \int d\mathrm{x}_{l+1}\, \frac{p(\mathrm{x}_{l+1}|\,\mathrm{x}_l)\; p(\mathrm{x}_l|\,\mathcal{Z}^l)}{\int d\mathrm{x}_l\; \underbrace{p(\mathrm{x}_{l+1}|\,\mathrm{x}_l)}_{\text{dynamics model}}\; \underbrace{p(\mathrm{x}_l|\,\mathcal{Z}^l)}_{\text{filtering } t_l}}\; \underbrace{p(\mathrm{x}_{l+1}|\,\mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

- $p(\mathrm{x}_{l+1}|\mathcal{Z}^k)$ retrodiction: last iteration step
- $p(\mathrm{x}_k|\mathrm{x}_{k-1})$ dynamic object behavior
- $p(\mathrm{x}_l|\mathcal{Z}^l)$ filtering at the time considered
- GAUSSians, GAUSSian mixtures: Exploit product formula!
- linear GAUSSian likelihood/dynamics: Rauch-Tung-Striebel smoothing

Exercise 4.3 Derive the *Rauch-Tung-Striebel* formulae

by using the Kalman filter assumptions

and the product formula (twice)!

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^{\top}\mathbf{P}_{l+1|l}^{-1}$$

$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^{\top}$$

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}_{k|k-1}^{-1}$$
'KALMAN gain matrix'

retrodiction: $\mathcal{N}(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k})$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}^\top$$

Exercise 4.4 Implement the *Rauch-Tung-Striebel* formulae in your simulator!

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$\begin{aligned} p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\ &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \end{aligned}$$

where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$\begin{aligned} p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\ p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\ &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big) \end{aligned}$$

Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(\mathbf{x}_l|\mathcal{Z}^k)$ and $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ based on the evolution model:

$$
\begin{aligned}
p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) &= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}, \mathbf{x}_{l+1}|\mathcal{Z}^k) \\
&= \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)
\end{aligned}
$$

where: $$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) = \frac{p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}{\int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)}$$

with: $$
\begin{aligned}
p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) &= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta},\, \mathbf{D}_{l+1|l+\theta}\big) \\
p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) &= \int d\mathbf{x}_l\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_l)\, p(\mathbf{x}_l|\mathcal{Z}^l) \\
p(\mathbf{x}_{l+1}|\mathcal{Z}^l) &= \int d\mathbf{x}_{l+\theta}\, p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \\
&= \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|l},\, \mathbf{P}_{l+1|l}\big)
\end{aligned}
$$

Looks like a Kalman filtering update!

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \qquad \text{Looks like filtering!}$$
$$= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}^{\top}_{l+\theta|l+1} \\
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^{\top}_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^{\top}_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \qquad \text{Looks like prediction!}$$

$$
\begin{aligned}
p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) &\propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \qquad \text{Looks like filtering!}\\
&= \mathcal{N}\big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)\\
&= \mathcal{N}\big(\mathbf{b}_{l+\theta|l+1};\, \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\big)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l})\\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1}\\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l}\\
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}^\top_{l+\theta|l+1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l}\\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}
$$

$$
p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \qquad \text{Looks like prediction!}
$$

$$p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta})\, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l)$$

Looks like filtering!

$$= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{a}_{l+\theta|l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big)$$
$$= \mathcal{N}\Big(\mathbf{b}_{l+\theta|l+1};\, \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1},\, \mathbf{\Delta}_{l+\theta|l+1}\Big)$$

$$\begin{aligned}
\mathbf{a}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+1|l+\theta}\mathbf{x}_{l+\theta|l}) \\
&= \mathbf{x}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1} \\
\mathbf{b}_{l+\theta|l+1} &= \mathbf{x}_{l+\theta} - \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\mathbf{x}_{l+1|l} \\
\mathbf{\Delta}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l} - \mathbf{\Phi}_{l+\theta|l+1}\mathbf{P}_{l+1|l}\mathbf{\Phi}^\top_{l+\theta|l+1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Phi}_{l+\theta|l+1} &= \mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta}\mathbf{P}^{-1}_{l+1|l} \\
\mathbf{P}_{l+1|l} &= \mathbf{F}_{l+1|l+\theta}\mathbf{P}_{l+\theta|l}\mathbf{F}^\top_{l+1|l+\theta} + \mathbf{D}_{l+1|l+\theta}.
\end{aligned}$$

$$p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k)\, p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$$

Looks like prediction!

$$= \mathcal{N}\Big(\mathbf{x}_{l+\theta};\, \mathbf{x}_{l+\theta|k},\, \mathbf{x}_{l+\theta|k}\Big)$$

$$\begin{aligned}
\mathbf{x}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\left(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}\right) \\
\mathbf{P}_{l+\theta|k} &= \mathbf{x}_{l+\theta|l} + \mathbf{\Phi}_{l+\theta|l+1}\left(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l}\right)\mathbf{\Phi}^\top_{l+\theta|l+1}
\end{aligned}$$

Kalman filter: linear Gaussian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\begin{aligned}\mathbf{x}_{k|k-1} &= \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}\\ \mathbf{P}_{k|k-1} &= \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}\end{aligned}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\begin{aligned}\mathbf{x}_{k|k} &= \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, & \boldsymbol{\nu}_{k|k-1} &= \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}\\ \mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, & \mathbf{S}_{k|k-1} &= \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k\\ & \mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} & & \text{'Kalman gain matrix'}\end{aligned}$$

retrodiction: $\mathcal{N}(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k})$

$$\begin{aligned}\mathbf{x}_{l|k} &= \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), & \mathbf{W}_{l|l+1} &= \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}\\ \mathbf{P}_{l|k} &= \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top\end{aligned}$$

Usually considered in tracking applications: pdfs of object states at given time instants

kinematic state: x_k, **sensor data accumulated over time:** $\mathcal{Z}^k = \{Z_k, \dots, Z_1\}$

Prediction:	$p(\mathrm{x}_{k-1}\|\mathcal{Z}^{k-1})$	$\xrightarrow[p(\mathrm{x}_k\|\mathrm{x}_{k-1})]{\text{dynamics model}}$	$p(\mathrm{x}_k\|\mathcal{Z}^{k-1})$
Filtering:	$p(\mathrm{x}_k\|\mathcal{Z}^{k-1})$	$\xrightarrow[\ell(\mathrm{x}_k;Z_k)]{\text{sensor model}}$	$p(\mathrm{x}_k\|\mathcal{Z}^k)$
Retrodiction:	$p(\mathrm{x}_l\|\mathcal{Z}^k)$	$\xleftarrow[\text{filtering}]{\text{dynamics model}}$	$p(\mathrm{x}_{l+1}\|\mathcal{Z}^k)$

The likelihood function $\ell(\mathrm{x}_k; Z_k)$ contains the full sensor information:

***current* measurements + *context* information on the sensor performance.**

$p(\mathrm{x}_k|\mathrm{x}_{k-1})$ contains context information on the object's kinematic properties.

In certain applications *accumulated* state vectors are to be considered.

Object state vectors accumulated over a time window $t_k, t_{k-1}, \dots, t_n$:

$$\mathbf{x}_{k:n} = (\mathbf{x}_k, \dots, \mathbf{x}_n)$$

The full information on $\mathbf{x}_{k:n}$ based on a time series of sensor data $\mathcal{Z}^k$ is contained in the corresponding *accumulated state density (ASD):*

$$p(\mathbf{x}_k, \dots, \mathbf{x}_n | \mathcal{Z}^k) = p(\mathbf{x}_{k:n} | \mathcal{Z}^k).$$

ASDs can be a useful notion in several applications. Examples are:

- **dealing with out-of-sequence measurements**
- **track-to-track fusion in sensor networks.**

Via marginalizing over all object states in the window, excepting $\mathbf{x}_l$, $n \leq l \leq k$,

$$\int \mathrm{d}\mathbf{x}_k, \ldots, \mathrm{d}\mathbf{x}_{l+1}, \mathrm{d}\mathbf{x}_{l-1}, \ldots, \mathrm{d}\mathbf{x}_n \, p(\mathbf{x}_k, \ldots, \mathbf{x}_n | \mathcal{Z}^k) = p(\mathbf{x}_l | \mathcal{Z}^k),$$

we obtain the pdfs for filtering & retrodiction; ASDs thus unify these notions.

In addition, all correlations between different instants of time are contained.

Bayes' Theorem: a recursion formula for updating ASDs:

$$p(\mathbf{x}_{k:n} | \mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k) \, p(\mathbf{x}_k | \mathbf{x}_{k-1}) \, p(\mathbf{x}_{k-1:n} | \mathcal{Z}^{k-1})}{\int \mathrm{d}\mathbf{x}_{k:n} \, \ell(\mathbf{x}_k; Z_k) \, p(\mathbf{x}_k | \mathbf{x}_{k-1}) \, p(\mathbf{x}_{k-1:n} | \mathcal{Z}^{k-1})}$$

A little formalistically speaking, 'sensor data processing' means:

Make sure that the sensor data are no longer explicitly present.

Closed-form Representations for ASDs

Example: conditions, where Kalman filtering is applicable

likelihood function: $\ell(\mathbf{x}_k; Z_k) = \mathcal{N}(\mathbf{z}_k;\ \mathbf{H}_k\mathbf{x}_k,\ \mathbf{R}_k)$

evolution model: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_k;\ \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}_{k|k-1})$

Induction argument (product formula, see paper on homepage):

The corresponding ASD is a Gaussian:

$$p(\mathbf{x}_{k:n}|Z^k) = \mathcal{N}(\mathbf{x}_{k:n};\ \mathbf{x}^k_{k:n},\ \mathbf{P}^k_{k:n})$$

with $\mathbf{x}^k_{k:n} = (\mathbf{x}^\top_{k|k}, \mathbf{x}^\top_{k-1|k}, \dots, \mathbf{x}^\top_{n|k})^\top$.

$\mathbf{x}_{k|k}$: **standard Kalman filtering,** $\mathbf{x}_{l|k}$, $l < k$: **standard RTS smoothing**

Fraunhofer FKIE

Block Matrix Representation of the ASD Covariance

$$\mathbf{P}^k_{k:n} =$$

$$\begin{pmatrix}
\mathbf{P}_{k|k} & \mathbf{P}_{k|k}\mathbf{W}^\top_{k-1|k} & \mathbf{P}_{k|k}\mathbf{W}^\top_{k-2|k} & \cdots & \mathbf{P}_{k|k}\mathbf{W}^\top_{1|k} \\
\mathbf{W}_{k-1|k}\mathbf{P}_{k|k} & \mathbf{P}_{k-1|k} & \mathbf{P}_{k-1|k}\mathbf{W}^\top_{k-2|k-1} & * & \mathbf{P}_{k-1|k}\mathbf{W}^\top_{1|k-1} \\
\mathbf{W}_{k-2|k}\mathbf{P}_{k|k} & \mathbf{W}_{k-2|k-1}\mathbf{P}_{k-1|k} & \mathbf{P}_{k-2|k} & * & \vdots \\
\vdots & * & * & * & \mathbf{P}_{2|k}\mathbf{W}^\top_{1|2} \\
\mathbf{W}_{1|k}\mathbf{P}_{k|k} & \mathbf{W}_{1|k-1}\mathbf{P}_{k-1|k} & \cdots & \mathbf{W}_{1|2}\mathbf{P}_{2|k} & \mathbf{P}_{1|k}
\end{pmatrix}$$

$\mathrm{P}_{k|k}$: standard Kalman filtering, $\mathrm{P}_{l|k}$, $l < k$: standard RTS smoothing

with: $\mathrm{W}_{l|k} = \prod_{\lambda=l}^{k-1} \mathrm{P}_{\lambda|\lambda}\mathbf{F}^\top_{\lambda+1|\lambda}\mathbf{P}^{-1}_{\lambda+1|\lambda}$ and $\mathrm{P}_{k+1|k}$: Kalman prediction.

Block Matrix Representations of the ASD Covariance

$$\mathbf{P}^k_{k:n} = \begin{pmatrix} \mathbf{P}_{k|k} & \mathbf{P}_{k|k}\mathbf{W}^\top_{k-1|k} & \mathbf{P}_{k|k}\mathbf{W}^\top_{k-2|k} & \cdots & \mathbf{P}_{k|k}\mathbf{W}^\top_{1|k} \\ \mathbf{W}_{k-1|k}\mathbf{P}_{k|k} & \mathbf{P}_{k-1|k} & \mathbf{P}_{k-1|k}\mathbf{W}^\top_{k-2|k-1} & * & \mathbf{P}_{k-1|k}\mathbf{W}^\top_{1|k-1} \\ \mathbf{W}_{k-2|k}\mathbf{P}_{k|k} & \mathbf{W}_{k-2|k-1}\mathbf{P}_{k-1|k} & \mathbf{P}_{k-2|k} & * & \vdots \\ \vdots & * & * & * & \mathbf{P}_{2|k}\mathbf{W}^\top_{1|2} \\ \mathbf{W}_{1|k}\mathbf{P}_{k|k} & \mathbf{W}_{1|k-1}\mathbf{P}_{k-1|k} & \cdots & \mathbf{W}_{1|2}\mathbf{P}_{2|k} & \mathbf{P}_{1|k} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{T}_{k|k} & -\mathbf{W}^\top_{k-1|k}\mathbf{Q}^{-1}_{k-1|k} & \mathbf{O} & \cdots & \mathbf{O} \\ -\mathbf{Q}^{-1}_{k-1|k}\mathbf{W}_{k-1|k} & \mathbf{T}_{k-1|k} & -\mathbf{W}^\top_{k-2|k}\mathbf{Q}^{-1}_{k-2|k} & \ddots & \vdots \\ \mathbf{O} & -\mathbf{Q}^{-1}_{k-2|k}\mathbf{W}_{k-2|k} & \ddots & \ddots & \mathbf{O} \\ \vdots & \ddots & \ddots & \mathbf{T}_{m+1|k} & -\mathbf{W}^\top_{m|k}\mathbf{Q}_{m|k} \\ \mathbf{O} & \cdots & \mathbf{O} & -\mathbf{Q}_{m|k}\mathbf{W}_{m|k} & \mathbf{T}_{m|k} \end{pmatrix}^{-1}$$

Markov property: tridiagonal structure of the inverse ASD covariance

Application to OoS Measurement Processing

Latencies in the communication infrastructure: ***Measurements can arrive at a processing node in a distributed data fusion system "too late", i.e. after sensor data with a time stamp newer than the time stamp of an out-of-sequence measurement have already been processed*** **[Seminal paper by Bar-Shalom, IEEE T-AES, 2002].**

In situations where Kalman filtering is applicable,

let z_m be produced at time t_m with $n \leq m < k$, and be arriving at time t_k.

With a projection matrix Π_m, defined by $\Pi_m \mathrm{x}_{k:n} = \mathrm{x}_m$,

the impact of z_m on the present and past object states is described by:

$$p(\mathrm{z}_m|\mathrm{x}_{k:n}) = \mathcal{N}(\mathrm{z}_m;\ \mathrm{H}_m \Pi_m \mathrm{x}_{k:n},\ \mathrm{R}_m).$$

Direct generalizations to situations with ambiguous sensor data / IMM models.

Standard Bayesian reasoning directly yields for the ASD:

$$p(\mathbf{x}_{k:n}|\mathbf{z}_m, Z^k) = \frac{p(\mathbf{z}_m|\mathbf{x}_{k:n})\, p(\mathbf{x}_{k:n}|Z^k)}{\int \mathrm{d}\mathbf{x}_{k:n}\, p(\mathbf{z}_m|\mathbf{x}_{k:n})\, p(\mathbf{x}_{k:n}|Z^k)}$$
$$= \mathcal{N}\Big(\mathbf{x}_{k:n};\, \mathbf{x}_{k:m:n},\, \mathbf{P}_{k:m:n}\Big)$$

with parameters obtained by a version of the Kalman update:

$$\mathbf{x}_{k:m:n} = \mathbf{x}_{k:n} + \mathbf{W}_{k:m:n}(\mathbf{z}_m - \mathbf{H}_m \mathbf{\Pi}_m \mathbf{x}_{k:n})$$
$$\mathbf{P}_{k:m:n} = \mathbf{P}_{k:n} - \mathbf{W}_{k:M:n} \mathbf{S}_{k,m,1} \mathbf{W}_{k:m:n}^\top$$

$$\mathbf{S}_{k:m:n} = \mathbf{H}_m \mathbf{\Pi}_m \mathbf{P}_{k:n} \mathbf{\Pi}_m^\top \mathbf{H}_m^\top + \mathbf{R}_m$$
$$\mathbf{W}_{k:m:n} = \mathbf{P}_{k:n} \mathbf{\Pi}_m^\top \mathbf{H}_m^\top \mathbf{S}_{k:m:n}^{-1}.$$

Note that $\mathrm{S}_{k:m:n}$ to be inverted has the same dimension as z_m.

Simultaneous update step for filtering and retrodiction

Efficient realizations exploit the structure of the ASD covariance.