Multiple Sensor Target Tracking: Basic Idea

Iterative updating of conditional probability densities!

kinematic target state x_k at time t_k, **accumulated sensor data** $\mathcal{Z}^k$

a priori knowledge: target dynamics models, sensor model

- **prediction:** $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \xrightarrow{\text{dynamics model}} p(\mathbf{x}_k|\mathcal{Z}^{k-1})$
- **filtering:** $p(\mathrm{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor model}]{\text{sensor data } Z_k} p(\mathrm{x}_k|\mathcal{Z}^k)$
- **retrodiction:** $p(\mathrm{x}_{l-1}|\mathcal{Z}^k) \xleftarrow[\text{dynamics model}]{\text{filtering output}} p(\mathrm{x}_l|\mathcal{Z}^k)$

The Multivariate GAUSSian Pdf

- *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$

- *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

  $q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

- *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\, \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top\mathbf{P}^{-1}(x-\mathbf{x})}$$

- GAUSS*ian Mixtures:* $p(x) = \sum_i p_i\, \mathcal{N}(x;\, \mathbf{x}_i,\, \mathbf{P}_i)$ (weighted sums)

Very First Look at an Important Data Fusion Algorithm

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

'KALMAN gain matrix'

A deeper look into the dynamics and sensor models necessary!

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Summary: BAYESian (Multi-) Sensor Tracking

- **Basis:** In the course of time one or several sensors produce measurements of targets of interest. Each target is characterized by its current state vector, being expected to change with time.

- **Objective:** Learn as much as possible about the individual target states at each time by analyzing the 'time series' which is constituted by the sensor data.

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- **Solution:** Derive iteration formulae for calculating the pdfs! Develop a mechanism for initiation! By doing so, exploit all background information available! Derive state estimates from the pdfs along with appropriate quality measures!

How to deal with probability density functions?

- pdf $p(x)$: Extract *probability statements* about the RV x by integration!
- naïvely: *positive* and *normalized* functions ($p(x) \geq 0$, $\int dx\, p(x) = 1$)
- *conditional pdf* $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on y on RV x?
- *marginal density* $p(x) = \int dy\, p(x,y) = \int dy\, p(x|y)\, p(y)$: Enter y!
- Bayes: $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx\, p(y|x)p(x)}$: $p(x|y) \leftarrow p(y|x)$, $p(x)$!

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- *certain knowledge* on x: $p(x) = \delta(x-y)$ '=' $\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-y)^2}{\sigma^2}}$

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- *transformed* $RV\, y = t[x]$: $p(y) = \int dx\, p(y,x) = \int dx\, p(y|x)\, p_x(x)$

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- *transformed RV* $y = t[x]$: $p(y) = \int dx\, p(y,x) = \int dx\, p(y|x)\, p_x(x) =$
  $\int dx\, \delta(y - t[x])\, p_x(x) =: [\mathcal{T} p_x](y)$ ($\mathcal{T}: p_x \mapsto p$, "transfer operator")

Characterize an object by *quantitatively describable* properties: object state

Examples:

- object position x on a strait line: $x \in \mathbb{R}$
- kinematic state $\mathbf{x} = (\mathbf{r}^\top, \dot{\mathbf{r}}^\top, \ddot{\mathbf{r}}^\top)^\top$, $\mathbf{x} \in \mathbb{R}^9$

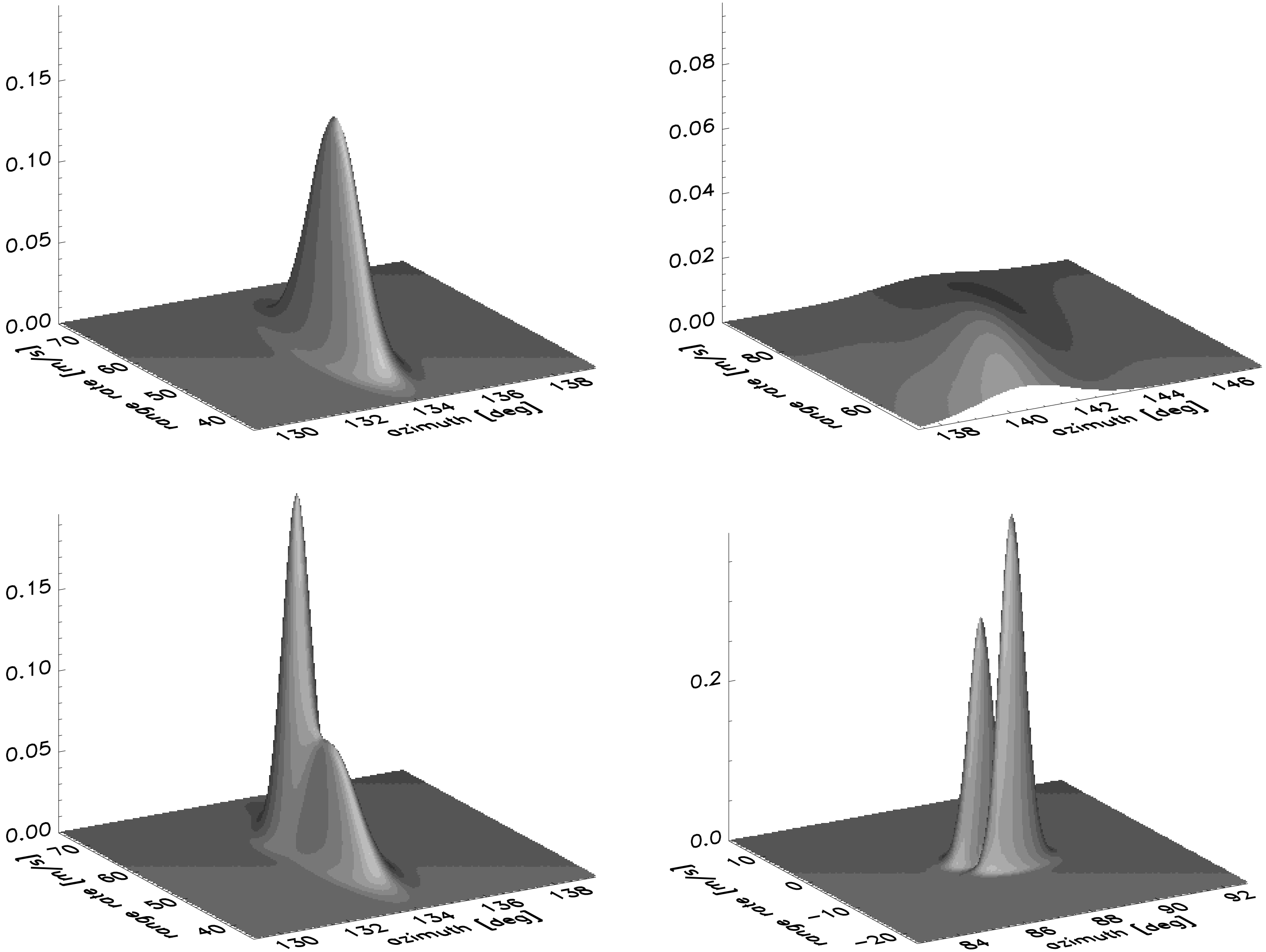
  position $\mathbf{r} = (x, y, z)^\top$, velocity $\dot{\mathbf{r}}$, acceleration $\ddot{\mathbf{r}}$
- joint state of two objects: $\mathbf{x} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top)^\top$
- kinematic state $\mathbf{x}$, object extension $\mathbf{X}$

  z.B. ellipsoid: symmetric, positively definite matrix
- kinematic state $\mathbf{x}$, object class class

  z.B. bird, sailing plane, helicopter, passenger jet, ...

Learn unknown object states from imperfect measurements and
describe by functions $p(\mathbf{x})$ imprecise knowledge mathematically precisely!

Interpret unknown object states as *random variables*, x [1D] or $\mathbf{x}$, $\mathbf{X}$ [vector / matrix variate]), characterized by corresponding *probability density functions* (pdf).



The concrete shape of the pdf $p(\mathbf{x})$ contains the full knowledge on $\mathbf{x}$!

Information on a random variable (RV) can be extracted by integration from the corresponding pdf !

at present: one dimensional case:

How probable is it that $x \in (a, b) \subseteq \mathbb{R}$ ***holds?***

Answer: $P\{x \in (a, b)\} = \int_a^b dx\, p(x) \qquad \Rightarrow \qquad p(x) \geq 0$

in particular: $P\{x \in \mathbb{R}\} = \int_{-\infty}^{\infty} dx\, p(x) = 1$ (normalzation)

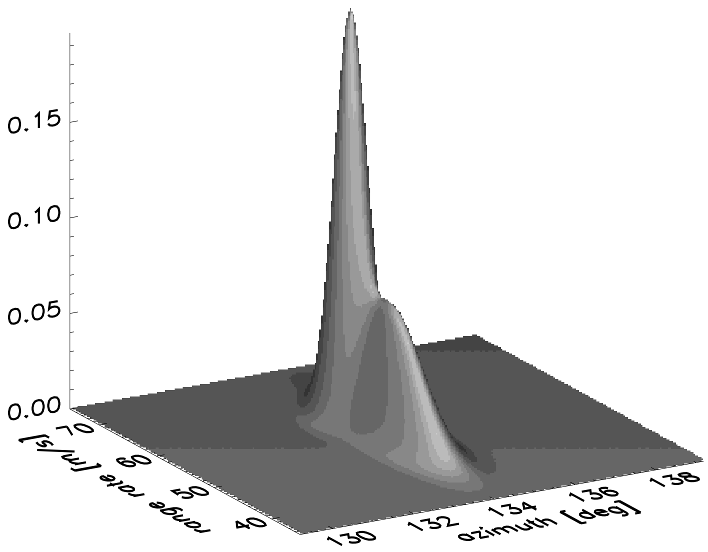
intuitive interpretation: “*the object is somewhere in* $\mathbb{R}$”

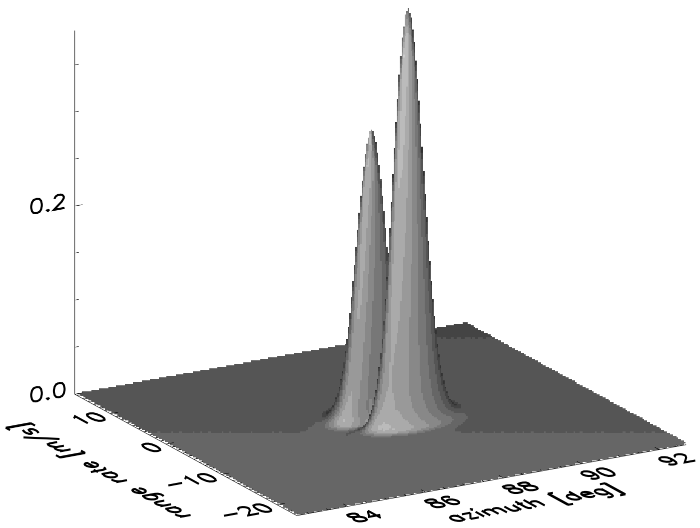
loosely: $p(x)\, dx$ **is probabity for** x **having a value between** x **and** $x + dx$

How to characterize the properties of a pdf?

specifically: How to associate a single “expected” value to a RV?

The maximum of the pdf is sometimes but not always useful!





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specifically: How to associate a single “expected” value to a RV?

The maximum of the pdf is sometimes but not always useful! ($\rightarrow$ examples)

instead: Calculate the centroid of the pdf!

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, p(x) = \bar{x} \qquad \text{“expectation value”}$$

How to characterize the properties of a pdf?

specifically: How to associate a single “expected” value to a RV?

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more generally: Consider functions $g : x \mapsto g(x)$ of the RV x!

$$\mathbb{E}[g(x)] = \int_{-\infty}^{\infty} dx\, g(x)\, p(x),$$ “expectation value of the *observable* g”

Example: Consider the observable $\frac{1}{2}mx^2$ (kinetic energy, x = speed)

An important observable: the “error” of an estimate

- **Quality:** How useful is an expectation value $\bar{x} = \mathbb{E}[x]$?

Consider special obervables as distance measure:

$$g(x) = |x - \bar{x}| \quad \text{oder} \quad g(x) = (x - \bar{x})^2$$

quadratic measures: computationally more comfortable!

‘expected error’ of the expectation value $\bar{x}$:

$$\mathbb{V}[x] = \mathbb{E}[(x - \bar{x})^2], \; \sigma_x = \sqrt{\mathbb{V}[x]}$$

variance, standard deviation

Exercise 3.1

Show that $\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$ holds.

Expectation value of the observable x^2 also called “2nd moment” of the pdf of x.

Calculate expectation and variance of the uniform density
of a RV $x \in \mathbb{R}$ in the intervall $[a, b]$.

Exercise 3.2

$$p(x) = \mathcal{U}(\underbrace{x}_{\text{ZV}};\ \underbrace{a, b}_{\text{Parameter}}) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

!

Pdf correctly normalized? $$\int_{-\infty}^{\infty} dx\, \mathcal{U}(x; a, b) = \frac{1}{b-a} \int_a^b dx = 1$$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{U}(x; a, b) = \frac{b+a}{2}$$

$$\mathbb{V}[x] = \frac{1}{b-a} \int_a^b dx\, x^2 - \mathbb{E}[x]^2 = \frac{1}{12}(b-a)^2$$

Important example: x *normally distributed* over $\mathbb{R}$ (Gauss)

- *wanted:* probabilities concentrated around μ
- quadratic distance: $||x-\mu||^2 = \frac{1}{2}(x-\mu)^2/\sigma^2$ (mathematically convenient!)
- Parameter σ is a measure of the "width" of the pdf: $||\sigma||^2 = \frac{1}{2}$
- for 'large' distances, i.e. $||x-\mu||^2 \gg \frac{1}{2}$, the pdf shall decay quickly.
- simplest approach: $\tilde{p}(x) = \mathrm{e}^{-||x-\mu||^2}$ $\quad$ ($> 0\ \forall x \in \mathbb{R}$, normalization?)
- Normalized for $p(x) = \tilde{p}(x) / \int_{-\infty}^{\infty} dx\, \tilde{p}(x)$!

Formula collection delivers: $\displaystyle\int_{-\infty}^{\infty} dx\, \tilde{p}(x) = \sqrt{2\pi}\sigma$

An admissible pdf with the required properties is obviously given by:

$$\mathcal{N}(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Exercise 3.3

Show for the Gauss density $p(x) = \mathcal{N}(x; \mu, \sigma)$:

$$\boxed{\mathbb{E}[x] = \mu, \;\; \mathbb{V}[x] = \sigma^2}$$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{N}(x; \mu, \sigma) = \mu$$

$$\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2$$

Use substitution and partial integration!

Use $\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\frac{1}{2}x^2} = \sqrt{2\pi}$!

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Remember your own ground truth generator!

Exercise 2.1

Consider an object that moves in two dimensions on the trajectory:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} \sin(\omega t) \\ \sin(2\omega t) \end{pmatrix} \quad \text{with} \quad A = \frac{v^2}{q}, \quad \omega = \frac{q}{2v}$$

and speed and acceleration parameters: $v = 300\,\frac{\text{m}}{\text{s}}$, $q = 9\,\frac{\text{m}}{\text{s}^2}$!

1. Plot the trajectory. Why is it periodical? What is its period $T = T(v, q)$?
2. Show for the velocity and acceleration vector:

$$\dot{\mathbf{r}}(t) = v \begin{pmatrix} \cos(\omega t)/2 \\ \cos(2\omega t) \end{pmatrix}, \quad \ddot{\mathbf{r}}(t) = -q \begin{pmatrix} \sin(\omega t)/4 \\ \sin(2\omega t) \end{pmatrix}!$$

3. Calculate for each instance of time t the tangential and normal vectors in $\mathbf{r}(t)$:

$$\mathbf{t}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}, \quad \mathbf{n}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \begin{pmatrix} -\dot{y}(t) \\ \dot{x}(t) \end{pmatrix}!$$

4. Plot $|\dot{\mathbf{r}}(t)|$, $|\ddot{\mathbf{r}}(t)|$, $\ddot{\mathbf{r}}(t)\mathbf{t}(t)$ and $\ddot{\mathbf{r}}(t)\mathbf{n}(t)$ over a period T!
5. Discuss the temporal behaviour based on the trajectory $\mathbf{r}(t)$!
6. What are the maximum speeds and accelerations, v_{max}, q_{max}?

Create your own sensor simulator!

Exercise 3.4

Simulate normally distributed (radar) measurements!

Measurement interval: ΔT = 5 s, sensor position: $\mathbf{r}_s$

State at time $t_k = k\Delta T$, $k \in \mathbb{Z}$: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$

1. Simulate measurements of the Cartesian position components of the target state $\mathbf{x}_k$:

$$\mathbf{z}_k^c = \begin{pmatrix} z_k^x \\ z_k^y \end{pmatrix} = \mathbf{H}\mathbf{x}_k + \mathbf{u}_k = (\,\mathsf{I}\;\mathsf{I}\;\mathsf{I}\,) \begin{pmatrix} \mathbf{r}_k \\ \dot{\mathbf{r}}_k \\ \ddot{\mathbf{r}}_k \end{pmatrix} + \sigma_c \begin{pmatrix} \text{normrnd}(0,1) \\ \text{normrnd}(0,1) \end{pmatrix}$$

   with a random number generator normrnd$(0, 1)$ producing normally distributed zero-mean and unit-variance random numbers, σ_c = 50 m denoting the standard deviation of the sensor measurement errors.

2. Simulate range and azimuth measurements of the target position $\mathbf{r}_k$:

$$\mathbf{z}_k^p = \begin{pmatrix} z_k^r \\ z_k^\varphi \end{pmatrix} = \begin{pmatrix} \sqrt{(x_k - x_s)^2 + (y_k - y_s)^2} \\ \arctan(\frac{y_k - y_s}{x_k - x_s}) \end{pmatrix} + \begin{pmatrix} \sigma_r \, \text{normrnd}(0,1) \\ \sigma_\varphi \, \text{normrnd}(0,1) \end{pmatrix}, \quad \mathbf{r}_{k,s} = (x_{k,s}, y_{k,s})^\top$$

   with σ_r = 20 m, σ_φ = 0.2° denoting the standard deviations in range and azimuth.

3. Plot the Cartesian and polar measurements $z_k^r(\cos z_k^\varphi, \sin z_k^\varphi)^\top + \mathbf{r}_s$ over the true target trajectory! Play with sensor positions and measurement error standard deviations!

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- *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

  $q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

- *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\ \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top \mathbf{P}^{-1}(x-\mathbf{x})}$$

Exercise 3.5 Show: $\int dx\ \mathrm{e}^{-q(x)} = \sqrt{|2\pi\mathbf{P}|}$, $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P}$

Trick: Symmetric, positively definite matrices can be diagonalized by an orthogonal coordinate transform.

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$$\mathbb{E}[x] = \mathbf{x}, \;\; \mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P} \;\; \text{(covariance)}$$

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$$\mathbb{E}[x] = \mathbf{x},\ \ \mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P} \ \ \text{(covariance)}$$

– *Covariance Matrix:* Expected error of the expectation.

A Useful Product Formula for GAUSSians

$$\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\,\mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Sketch of the proof:

- Interpret $\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\,\mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P})$ as a joint pdf $p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{z}, \mathbf{x})$.
- Show that $p(\mathbf{z}, \mathbf{x})$ is a GAUSSian: $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\left(\left(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\right);\, \left(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\right),\, \left(\begin{smallmatrix}\mathbf{S} & \mathbf{H}\mathbf{P}\\ \mathbf{P}\mathbf{H}^\top & \mathbf{P}\end{smallmatrix}\right)\right)$.
- Calculate from $p(\mathbf{z}, \mathbf{x})$ the marginal and conditional pdfs $p(\mathbf{z})$ and $p(\mathbf{x}|\mathbf{z})$.
- From $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) = p(\mathbf{x}, \mathbf{z})$ we obtain the result.

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\big(\mathbf{x};\, \mathbb{E}[\mathbf{x}],\, \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\, \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\, \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^{\top}\big)$$

Affine Transforms of Gaussian Random Variables

$$\boxed{\mathcal{N}\big(\mathbf{x};\,\mathbb{E}[\mathbf{x}],\,\mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\,\mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\,\mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)}$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

Affine Transforms of Gaussian Random Variables

$$\mathcal{N}\big(\mathbf{x};\,\mathbb{E}[\mathbf{x}],\,\mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\,\mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\,\mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\,\mathbf{y},\,\mathbf{D}\big)$ with $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D} \to 0$$

Affine Transforms of GAUSSian Random Variables

$$\boxed{\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)}$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{D}\big)$ with $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D} \to 0$$

$$= \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top + \mathbf{D}\big) \qquad \text{for } \mathbf{D} \to 0; \quad \text{product formula!}$$

Also true if $\text{dim}(\mathbf{x}) \neq \text{dim}(\mathbf{y})$**!**

A popular model for object evolutions

Piecewise Constant White Acceleration Model

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

For known $\mathbf{x}_{k-1}$ and without external influences we have with $\Delta T_k = t_k - t_{k-1}$:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},$$ **see blackboard!**

Assume during the interval ΔT_k a constant acceleration $\mathbf{a}_k$ causing the state evolution:

$$\begin{pmatrix} \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \Delta T_k\,\mathbf{I} \end{pmatrix} \mathbf{a}_k =: \mathbf{G}_k\mathbf{a}_k,$$ **linear transform!**

Let $\mathbf{a}_k$ be a Gaussian RV with pdf: $p(\mathbf{a}_k) = \mathcal{N}\big(\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2\mathbf{I}\big)$, we therefore have:

$$p(\mathbf{G}_k\mathbf{a}_k) = \mathcal{N}\big(\mathbf{G}_k\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2\mathbf{G}_k\mathbf{G}_k^\top\big).$$

Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\Big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\Big)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\, \mathbf{I} & \frac{1}{2}\Delta T_k^3\, \mathbf{I} \\ \frac{1}{2}\Delta T_k^3\, \mathbf{I} & \Delta T_k^2\, \mathbf{I} \end{pmatrix}$$

Kalman filter: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$
'KALMAN gain matrix'

Exercise 3.6

In your sensor simulator, chose a sensor at position $\mathbf{r}_s$, for example $\mathbf{r}_s = (0, 0)^\top$, that produces measurements $\mathbf{z}_k$ of the Cartesian target positions $\mathbf{H}\mathbf{x}_k$ from your ground truth generator. Use the measurement covariance matrix $\mathbf{R} = \sigma_c^2\mathrm{diag}[1, 1]$, σ_c = 50 m, for all measurements, but allow individual measurement error covariances for each measurement. Program your first Kalman filter using a constant acceleration. Visualize your results nicely! Compare the ground truth, the measurement, and the estimates!

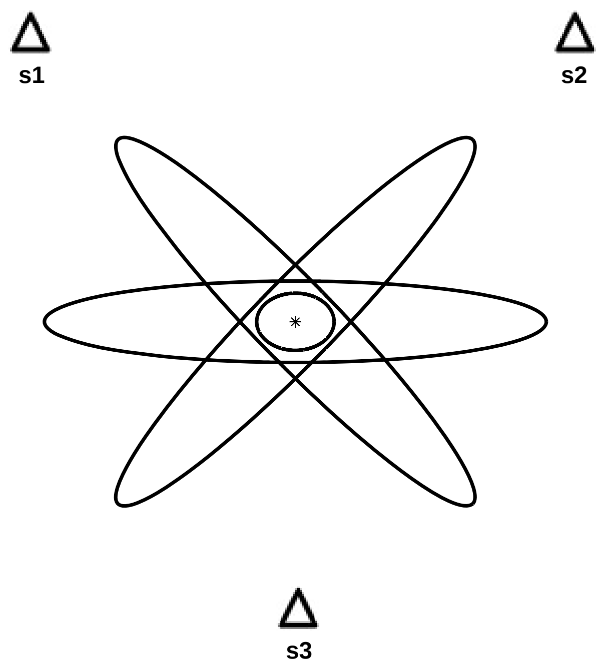
Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

a closer look: The error of each measurement $\mathbf{z}_i$ is described by a related measurement error *covariance matrix* $\mathbf{R}_i$ ('error ellipsoids'). In 2 dimensions:



$\mathbf{R}_i$ can strongly depend on the underlying senor-to-target geometry!

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

  $\mathbf{z}_k = (r_k, \varphi_k)^\top$, measurement error: $\mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}$, r, φ independent

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

- **What is the likelihood function in Cartesian coordinates?**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix}$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

constant and linear term of a Taylor series only, blackboard!

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \quad r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **affine transform of GAUSSian random variables:**

$$\mathcal{N}\big(\mathbf{z};\, \mathbf{x},\, \mathbf{R}\big) \xrightarrow{\mathbf{z}' = \mathbf{t} + \mathbf{T}\mathbf{z}} \mathcal{N}\big(\mathbf{z}';\, \mathbf{t} + \mathbf{T}\mathbf{x},\, \mathbf{T}\mathbf{R}\mathbf{T}^\top\big)$$

More Realistic: Range, Azimuth Measurements

- **measurements in polar coordinates:**

$$\mathbf{z}_k = (r_k, \varphi_k)^\top, \quad \text{measurement error: } \mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}, \; r, \varphi \text{ independent}$$

- **in Cartesian coord.: expand around** $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$**:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

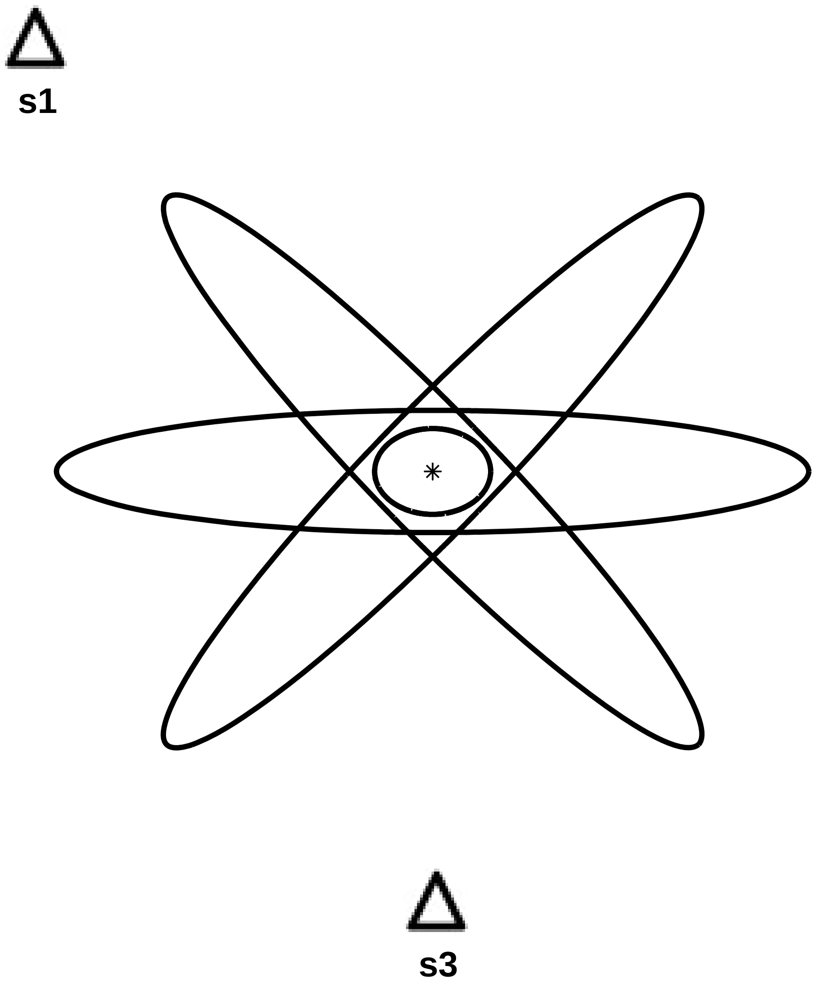
$$\mathbf{T} = \frac{\partial \mathbf{t}[\mathbf{r}_{k|k-1}]}{\partial \mathbf{r}_{k|k-1}} = \begin{pmatrix} \cos\varphi_{k|k-1} & -r_{k|k-1}\sin\varphi_{k|k-1} \\ \sin\varphi_{k|k-1} & r_{k|k-1}\cos\varphi_{k|k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{\text{rotation } \mathbf{D}_\varphi} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{dilation } \mathbf{S}_r}$$

- **Cartesian error covariance (time dependent):**

$$\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\, \mathbf{R}\, \mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top$$

- **sensor fusion: sensor-to-target-geometry enters into** $\mathbf{T}\mathbf{R}\mathbf{T}^\top$

Fraunhofer FKIE



sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathsf{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$p(\mathbf{z}_k^1, \mathbf{z}_k^2|\mathbf{x}_k) = p(\mathbf{z}_k^1|\mathbf{x}_k)\, p(\mathbf{z}_k^2|\mathbf{x}_k) \qquad \text{independent sensors}$$

$$= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big)$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $$\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{S_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, S_k$:

$$p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) = p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \qquad \text{independent sensors}$$

$$= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big)$$

A Useful Product Formula for GAUSSians

$$\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\, \mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

S_k Sensors Producing Target Measurement at the Same Time

One possibility: $\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{S_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \dots, \mathbf{R}_k^{S_k}]$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \dots, S_k$:

$$
\begin{aligned}
p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) &= p(\mathbf{z}_k^1|\mathbf{x}_k)\, p(\mathbf{z}_k^2|\mathbf{x}_k) \qquad \text{independent sensors} \\
&= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \\
&= \mathcal{N}\big(\mathbf{H}_k\mathbf{x}_k;\, \mathbf{z}_k^1,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \\
&\propto \mathcal{N}\big(\mathbf{H}_k\mathbf{x}_k;\, \underbrace{\mathbf{R}_k(\mathbf{R}_k^{1\,-1}\mathbf{z}_k^1 + \mathbf{R}_k^{2\,-1}\mathbf{z}_k^2)}_{=\mathbf{z}_k},\, \underbrace{(\mathbf{R}_k^{1\,-1} + \mathbf{R}_k^{2\,-1})^{-1}}_{=\mathbf{R}_k}\big) \\
&= \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)
\end{aligned}
$$

Exercise 3.7 Generalize to the case $S_k > 2$ (induction argument)!

One possible fusion strategy: Create a singe *effective measurement* by preprocessing of the individual measurements!

$$\mathbf{z}_k = \mathbf{R}_k \sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1} \mathbf{z}_k^s$$

weighted arithmetic mean of measurements

$$\mathbf{R}_k = \left(\sum_{s=1}^{S_k} \left(\mathbf{R}_k^s\right)^{-1}\right)^{-1}$$

harmonic mean of measuremt covariances

A typical structure for fusion equations!

With measurement specific measurement error covariances, your Kalman filter already is a multiple sensor fusion algorithms. Try and play!