Exercise 3.1

Zeige, dass gilt: $\boxed{\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2}$!

The expectation of the obvervable x^2 is also called “2nd Moment” dof the pdf of x.

Exercise 3.1

Show: $\boxed{\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2}$!

mit $\bar{x} = \mathbb{E}[x]$ gilt: $\mathbb{V}[x] = \int_{-\infty}^{\infty} dx \; (x - \bar{x})^2 \, p(x)$

Exercise 3.1

Show: $\boxed{\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2}$!

mit $\bar{x} = \mathbb{E}[x]$ gilt: $\mathbb{V}[x] = \int_{-\infty}^{\infty} dx\, (x - \bar{x})^2\, p(x)$

$$= \int_{-\infty}^{\infty} dx\, x^2\, p(x) - 2\bar{x} \int_{-\infty}^{\infty} dx\, x\, p(x) + \bar{x}^2 \int_{-\infty}^{\infty} dx\, p(x)$$

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$$= \underbrace{\int_{-\infty}^{\infty} dx\, x^2\, p(x)}_{=\mathbb{E}[x^2]} - 2\bar{x} \underbrace{\int_{-\infty}^{\infty} dx\, x\, p(x)}_{=\bar{x} \text{ Erwartung}} + \bar{x}^2 \underbrace{\int_{-\infty}^{\infty} dx\, p(x)}_{=1 \text{ Normierung}}$$

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$$= \mathbb{E}[x^2] - 2\bar{x}^2 + \bar{x}^2 = \mathbb{E}[x^2] - \mathbb{E}[x]^2 \qquad \diamond$$

The expectation of the obvervable x^2 is also called “2nd Moment” dof the pdf of x.

Exercise 3.2

Calculate expectation and variance of the uniform density of a RV $x \in \mathbb{R}$ in the intervall $[a, b]$

$$p(x) = \mathcal{U}(\underbrace{x}_{\text{ZV}}; \underbrace{a, b}_{\text{Parameter}}) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

!

Normalized? $\int_{-\infty}^{\infty} dx\, \mathcal{U}(x; a, b) = 1$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{U}(x; a, b) = \frac{b + a}{2}$$

$$\mathbb{V}[x] = \frac{1}{b - a} \int_{a}^{b} dx\, x^2 - \mathbb{E}[x]^2 = \frac{1}{12}(b - a)^2$$

Calculate expectation and variance of the uniform density

Exercise 3.2

$$p(x) = \mathcal{U}(x; a, b) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$!

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$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{U}(x; a, b) = \frac{1}{b-a} \int_a^b dx\, x$$

Calculate expectation and variance of the uniform density

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$$p(x) = \mathcal{U}(x; a, b) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

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$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{U}(x; a, b) = \frac{1}{b-a} \int_a^b dx\, x = \frac{\frac{1}{2} x^2 |_a^b}{b-a} = \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{b+a}{2}$$

Calculate expectation and variance of the uniform density

Exercise 3.2

$$p(x) = \mathcal{U}(x; a, b) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$!

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$$\mathbb{V}[x] = \frac{1}{b-a} \int_a^b dx\, x^2 - \mathbb{E}[x]^2$$ Exercise 3.1!

Calculate expectation and variance of the uniform density

Exercise 3.2

$$p(x) = \mathcal{U}(x; a, b) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

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$$\mathbb{V}[x] = \frac{1}{b-a} \int_a^b dx\, x^2 - \mathbb{E}[x]^2 = \frac{1}{3}\frac{b^3 - a^3}{b-a} - \frac{1}{4}(b+a)^2$$

Calculate expectation and variance of the uniform density

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$$= \frac{4b^3 - 4a^3 - 3b^3 - 3ab^2 + 3a^2 b + 3a^3}{12(b-a)}$$

Calculate expectation and variance of the uniform density

Exercise 3.2

$$p(x) = \mathcal{U}(x; a, b) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

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Calculate expectation and variance of the uniform density

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$$p(x) = \mathcal{U}(x; a, b) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{sonst} \end{cases}$$

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$$= \frac{4b^3 - 4a^3 - 3b^3 - 3ab^2 + 3a^2b + 3a^3}{12(b-a)} = \frac{(b-a)^3}{12(b-a)} = \frac{1}{12}(b-a)^2$$

Exercise 2.3

Show for the Gauss density $p(x) = \mathcal{N}(x; \mu, \sigma)$:

$$\boxed{\mathbb{E}[x] = \mu,\ \mathbb{V}[x] = \sigma^2}$$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{N}(x; \mu, \sigma) = \mu$$

$$\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2$$

Solution via substitution and partielle integration!

Use $\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\frac{1}{2}x^2} = \sqrt{2\pi}$!

$$\mathbb{E}[x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\, x\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma dy$$

$$
\begin{aligned}
\mathbb{E}[x] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\, x\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, (\sigma y + \mu)\, \mathrm{e}^{-\frac{1}{2}y^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[x] &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\, x\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, (\sigma y + \mu)\, \mathrm{e}^{-\frac{1}{2}y^2} \\
&= \frac{\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} dy\, y\, \mathrm{e}^{-\frac{1}{2}y^2} - \frac{\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} dy\, y\, \mathrm{e}^{-\frac{1}{2}y^2} + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\frac{1}{2}y^2} = \mu
\end{aligned}
$$

$$\mathbb{E}[x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\, x\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, (\sigma y + \mu)\, \mathrm{e}^{-\frac{1}{2}y^2}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} dy\, y\, \mathrm{e}^{-\frac{1}{2}y^2} - \frac{\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} dy\, y\, \mathrm{e}^{-\frac{1}{2}y^2} + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\frac{1}{2}y^2} = \mu$$

$$\mathbb{E}[x^2] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\, x\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, (\sigma y + \mu)^2\, \mathrm{e}^{-\frac{1}{2}y^2}$$

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$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, y^2\, \mathrm{e}^{-\frac{1}{2}y^2} + \mu^2, \quad \mathbb{V}[x] = \mathbb{E}[x^2] - \big(\mathbb{E}[x]\big)^2$$

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$$= \frac{\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} dy\, y\, \mathrm{e}^{-\frac{1}{2}y^2} - \frac{\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} dy\, y\, \mathrm{e}^{-\frac{1}{2}y^2} + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\frac{1}{2}y^2} = \mu$$

$$\mathbb{E}[x^2] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\, x\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, (\sigma y + \mu)^2\, \mathrm{e}^{-\frac{1}{2}y^2}$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, y^2\, \mathrm{e}^{-\frac{1}{2}y^2} + \mu^2, \quad \left(\mathrm{e}^{-\frac{1}{2}y^2}\right)' = -y\, \mathrm{e}^{-\frac{1}{2}y^2}$$

partial integration: $\int_a^b dx\, g(x) f'(x) = g(x) f(x)\Big|_a^b - \int_a^b dx\, g'(x) f(x)$

$$\mathbb{E}[x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} dx\, x\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad y = \frac{x-\mu}{\sigma}, \quad \frac{dy}{dx} = \frac{1}{\sigma}, \quad x = \sigma y + \mu, \quad dx = \sigma dy$$

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partial integration: $\int_a^b dx\, g(x) f'(x) = g(x) f(x)\Big|_a^b - \int_a^b dx\, g'(x) f(x)$

$$\mathbb{V}[x^2] = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, y \left(y\, \mathrm{e}^{-\frac{1}{2}y^2}\right) = \frac{-\sigma^2}{\sqrt{2\pi}} \underbrace{\left(y\, \mathrm{e}^{-\frac{1}{2}y^2}\right)\Big|_{-\infty}^{\infty}}_{\to 0 \ \text{ symmetry!}} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-\frac{1}{2}y^2} = \sigma^2$$