The Four Columns of Sensor Data Fusion

- **Statistical Estimation Theory: Object States**
- **Combinatorial Optimization: Data Association**
- **Optimal Decision Making: Track Initiation**
- **Resources Management**

Many multifunctional and / or multiple sensor systems make use of these distinctions. More innovative approaches develop a unified methodology.

Kalman filter: general properties

$$
\begin{aligned}
\mathbf{x}_{k|k} &= \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\, \boldsymbol{\nu}_{k|k-1} \\
\mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}^{\top}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}_{k|k-1} &= \mathbf{H}_{k|k-1}\mathbf{P}_{k|k-1}\mathbf{H}_{k|k-1}^{\top} + \mathbf{R}_{k|k-1} \\
\mathbf{W}_{k|k-1} &= \mathbf{P}_{k|k-1}\mathbf{H}_{k|k-1}^{\top}\mathbf{S}_{k|k-1}
\end{aligned}
$$

- *all elements* of the density: estimate an quality measure
- *variable* update, *time*-dependent evolution, measurement error
- variable *type*: measurement matrix (e.g. incomplete measurements)
- low computational effort (e.g. analytic inversions)

Multiple sensors producing n_k data at the same time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{n_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \text{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{n_k}]$$

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Multiple sensors producing n_k data at the same time

One possibility: $$\mathbf{H}_k\mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{n_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{n_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, n_k$:

$$\begin{aligned} p(\mathbf{z}_k^1, \mathbf{z}_k^2|\mathbf{x}_k) &= p(\mathbf{z}_k^1|\mathbf{x}_k)\, p(\mathbf{z}_k^2|\mathbf{x}_k) \\ &= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k^2\big) \end{aligned}$$

Multiple sensors producing n_k data at the same time

One possibility: $\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{n_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{n_k}]$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, n_k$:

$$\begin{aligned}
p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) &= p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \\
&= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big) \\
&= \mathcal{N}\big(\mathbf{H}_k \mathbf{x}_k;\, \mathbf{z}_k^1,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big) \\
&\propto \mathcal{N}\Big(\mathbf{H}_k \mathbf{x}_k;\, \underbrace{\mathbf{R}_k (\mathbf{R}_k^{1\ -1} \mathbf{z}_k^1 + \mathbf{R}_k^{2\ -1} \mathbf{z}_k^2)}_{=\mathbf{z}_k},\, \underbrace{(\mathbf{R}_k^{1\ -1} + \mathbf{R}_k^{2\ -1})^{-1}}_{=\mathbf{R}_k}\Big) \\
&= \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k\big)
\end{aligned}$$

Multiple sensors producing n_k data at the same time

One possibility: $$\mathbf{H}_k \mathbf{x}_k = \begin{pmatrix} \mathbf{H}_k^1 \\ \vdots \\ \mathbf{H}_k^{n_k} \end{pmatrix} \mathbf{x}_k, \quad \mathbf{R}_k = \mathrm{diag}[\mathbf{R}_k^1, \ldots, \mathbf{R}_k^{n_k}]$$

Alternatively, provided that $\mathbf{H}_k^i = \mathbf{H}_k$, $i = 1, \ldots, n_k$:

$$\begin{aligned} p(\mathbf{z}_k^1, \mathbf{z}_k^2 | \mathbf{x}_k) &= p(\mathbf{z}_k^1 | \mathbf{x}_k)\, p(\mathbf{z}_k^2 | \mathbf{x}_k) \\ &= \mathcal{N}\big(\mathbf{z}_k^1;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big) \\ &= \mathcal{N}\big(\mathbf{H}_k \mathbf{x}_k;\, \mathbf{z}_k^1,\, \mathbf{R}_k^1\big)\, \mathcal{N}\big(\mathbf{z}_k^2;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k^2\big) \\ &\propto \mathcal{N}\Big(\mathbf{H}_k \mathbf{x}_k;\, \underbrace{\mathbf{R}_k (\mathbf{R}_k^{1\,-1} \mathbf{z}_k^1 + \mathbf{R}_k^{2\,-1} \mathbf{z}_k^2)}_{= \mathbf{z}_k},\, \underbrace{(\mathbf{R}_k^{1\,-1} + \mathbf{R}_k^{2\,-1})^{-1}}_{= \mathbf{R}_k}\Big) \\ &= \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k\big) \end{aligned}$$

Exercise 8.1 (voluntary) Generalize to the case $n_k > 2$!

Multiple Hypothesis Tracking: Basic Idea

Iterative updating of conditional probability densities!

kinematic target state x_k at time t_k, **accumulated sensor data** $\mathcal{Z}^k$

a priori knowledge: target dynamics models, sensor model, road maps

• **prediction:**	$p(\mathbf{x}_{k-1}\|\mathcal{Z}^{k-1})$	$\xrightarrow[\text{road maps}]{\text{dynamics model}}$	$p(\mathbf{x}_k\|\mathcal{Z}^{k-1})$
• **filtering:**	$p(\mathrm{x}_k\|\mathcal{Z}^{k-1})$	$\xrightarrow[\text{sensor model}]{\text{sensor data } Z_k}$	$p(\mathrm{x}_k\|\mathcal{Z}^k)$
• **retrodiction:**	$p(\mathbf{x}_{l-1}\|\mathcal{Z}^k)$	$\xleftarrow[\text{dynamics model}]{\text{filtering output}}$	$p(\mathbf{x}_l\|\mathcal{Z}^k)$

– ***finite mixture:*** **inherent ambiguity (data, model, road *network*)**
– ***optimal estimators:*** **e.g. minimum mean squared error (MMSE)**
– ***initiation of pdf iteration:*** **multiple hypothesis track extraction**

- $p(\mathbf{x}_k|\mathcal{Z}^{k-1})$ is a *prediction* of the target state at time t_k based on all measurements in the *past*.

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, \mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \qquad \text{marginal pdf}$$

$$= \int d\mathbf{x}_{k-1}\, \underbrace{p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1})}_{\text{object dynamics!}}\, \underbrace{p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}_{\text{idea: iteration!}} \qquad \text{notion of a conditional pdf}$$

often: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1})$ (MARKOV)

sometimes: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}}_{\text{deterministic}}\mathbf{x}_{k-1},\, \underbrace{\mathbf{D}_{k|k-1}}_{\text{random}}\big)$ (linear GAUSS-MARKOV)

- $p(Z_k, m_k|\mathbf{x}_k) \propto \ell(\mathbf{x}_k; Z_k, m_k)$ describes, what the *current* sensor output Z_k, m_k can say about the current target state $\mathbf{x}_k$ and is called *likelihood function*.

$\ell(Z_k, m_k; \mathbf{x}_k) = (1 - P_D)\rho_F + P_D \sum_{j=1}^{m_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}_k^j\mathbf{x}_k,\, \mathbf{R}_k^j\big)$ (1 target, m_k measurements)

iteration formula: $$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}$$

GAUSSian transition pdf: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1})$

with: $\underbrace{\mathbf{F}_{k|k-1} \text{ (evolution matrix)}}_{\text{describes deterministic motion}}$, $\underbrace{\mathbf{D}_{k|k-1} \text{ (dynamics covariance matrix)}}_{\text{models of random maneuvers}}$

GAUSSian posterior: $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1})$

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \int d\mathbf{x}_{k-1}\, \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)}_{\text{dynamics model}}\, \underbrace{\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big)}_{\text{posterior at time } t_{k-1}}$$

$$= \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}}_{=:\mathbf{x}_{k|k-1}},\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\top}_{k|k-1} + \mathbf{D}_{k|k-1}}_{=:\mathbf{P}_{k|k-1}}\big)$$

$$\times \underbrace{\int d\mathbf{x}_{k-1}\, \mathcal{N}\big(\mathbf{x}_{k-1};\, \ldots,\, \ldots\big)}_{=1 \quad \text{(normalization!)}} \qquad \text{(exploit product formula!)}$$

$$= \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$$

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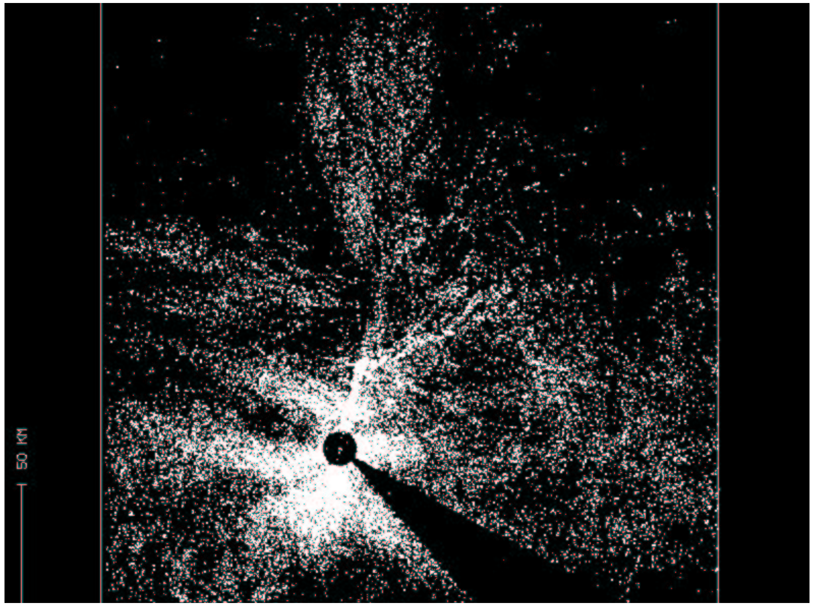

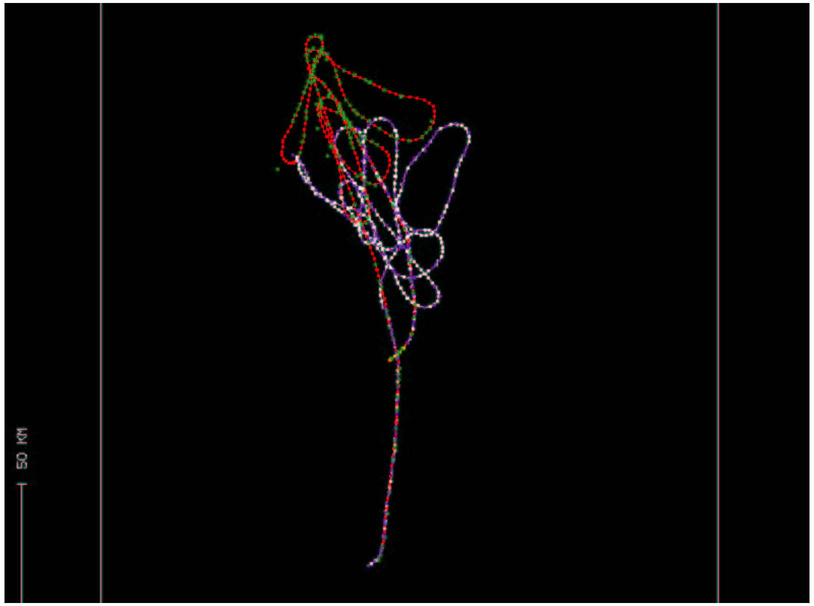

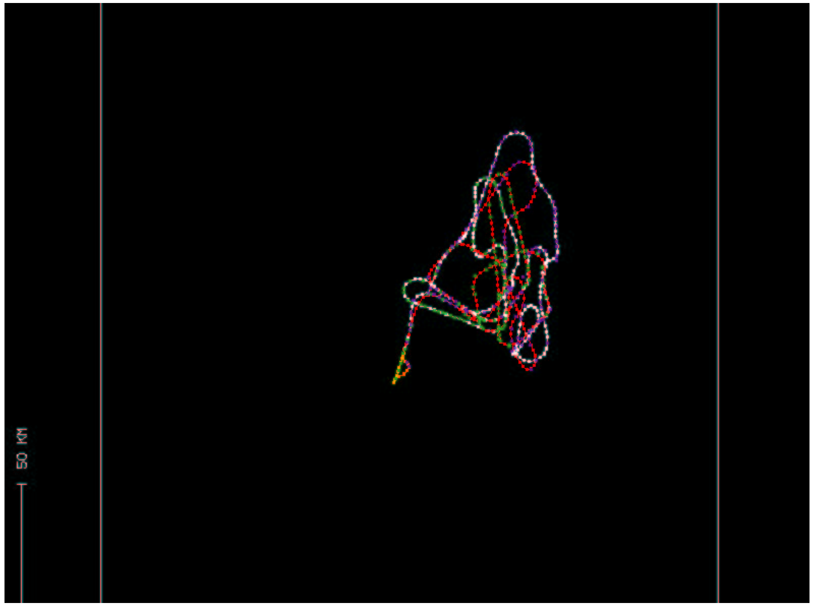

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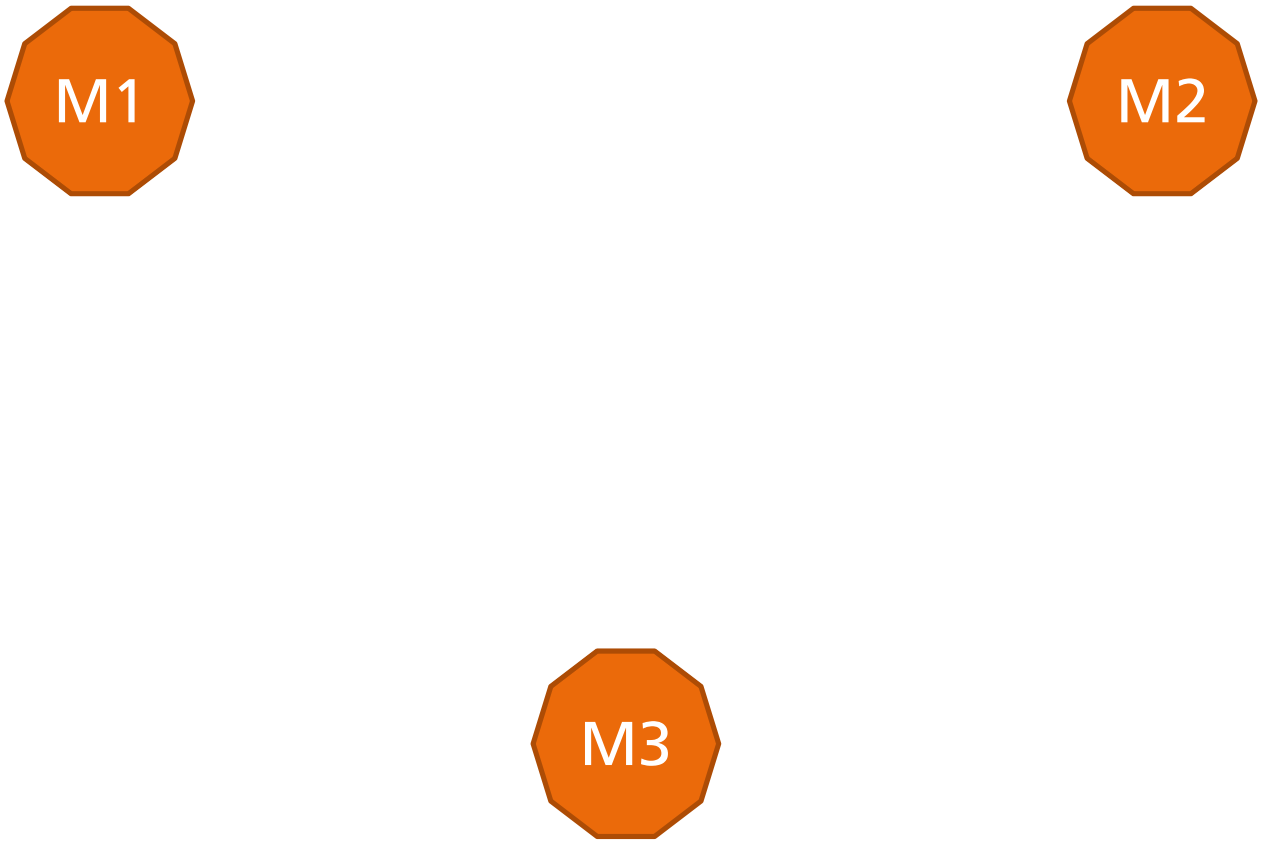
in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

Quite general: agent switching between different modes of over-all behavior



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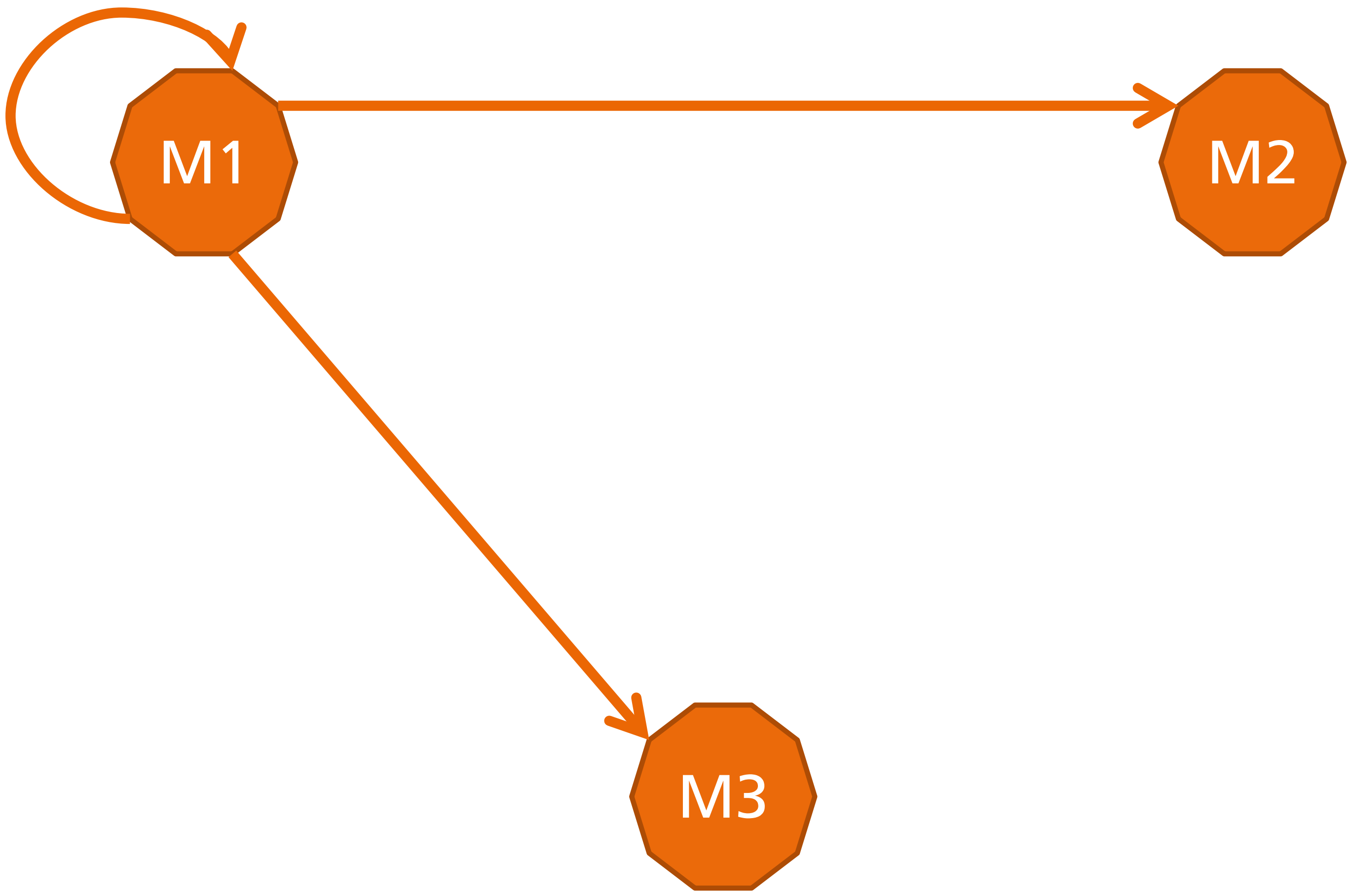
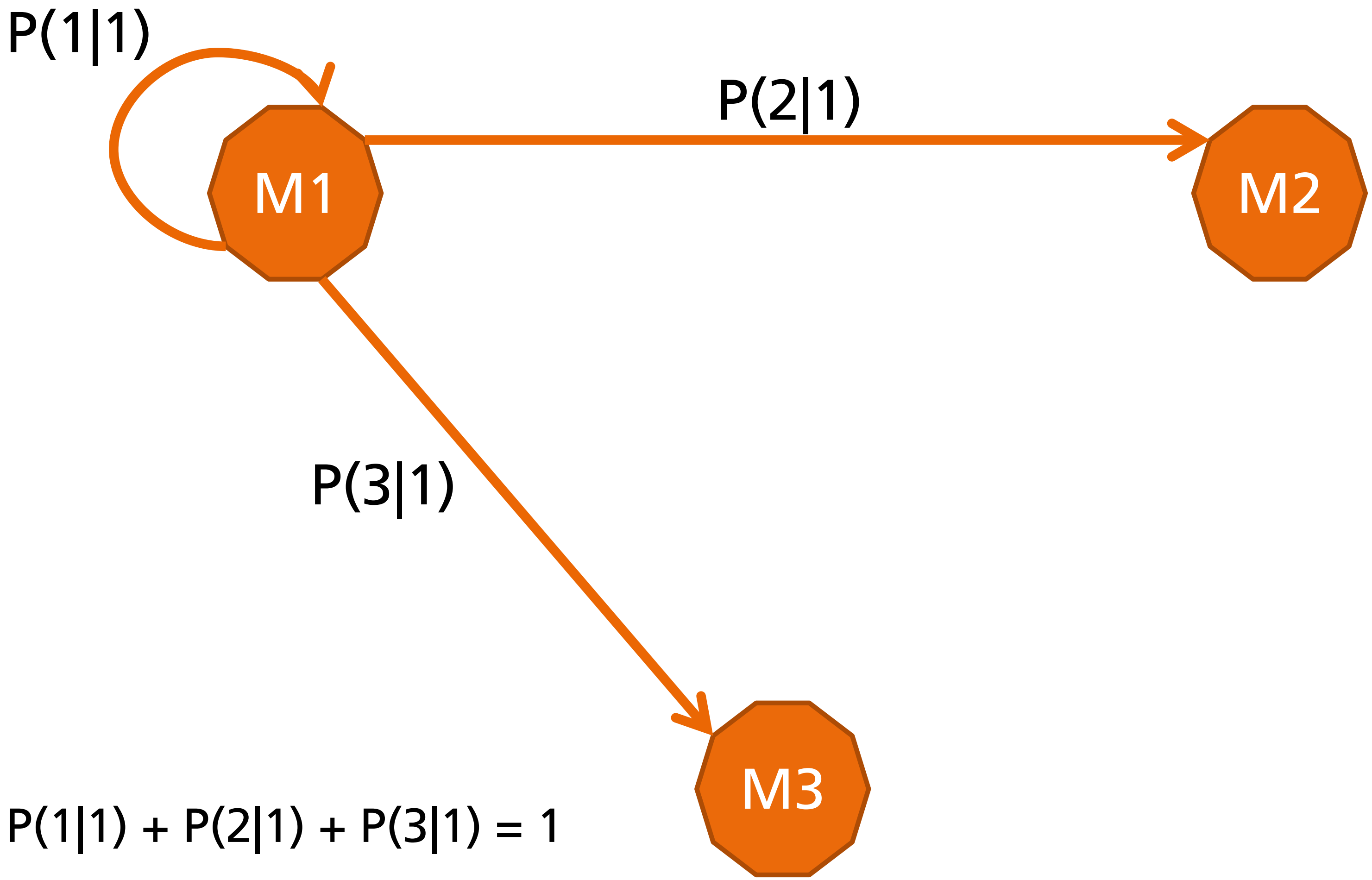
Quite general: agent switching between different modes of over-all behavior



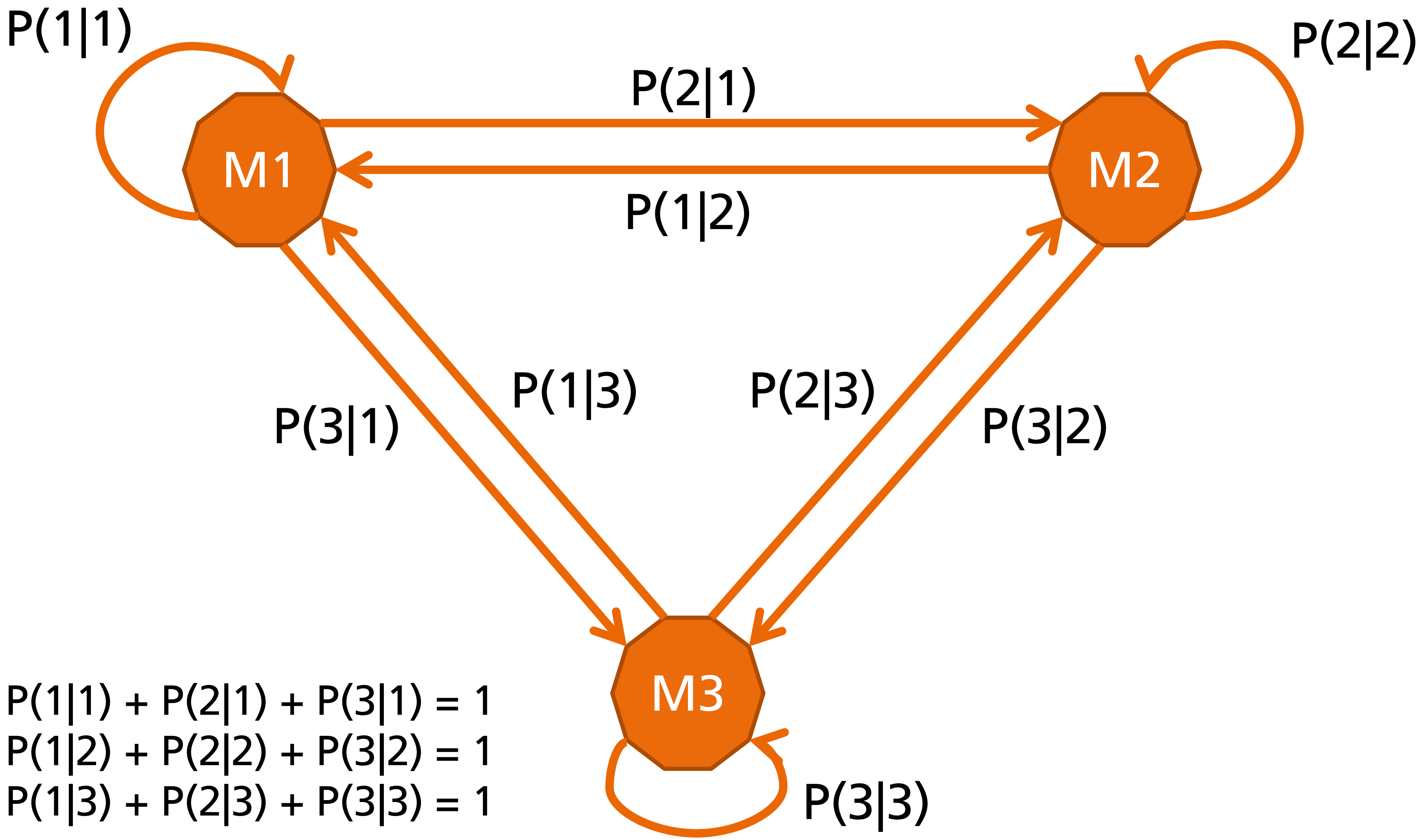
Quite general: agent switching between different modes of over-all behavior



$P(1|1) + P(2|1) + P(3|1) = 1$

Fraunhofer
FKIE

Quite general: agent switching between different modes of over-all behavior



$$P(1|1) + P(2|1) + P(3|1) = 1$$
$$P(1|2) + P(2|2) + P(3|2) = 1$$
$$P(1|3) + P(2|3) + P(3|3) = 1$$

A quite general mathematical structure: a graph, characterized by

nodes (here: evolution models) and directed edges defining an adjacency matrix

(here: transition matrix P, stochastic matrix: columns sum up to one)

initial information on which model is currently being in effect: $\mathbf{p}_k = (p_k^1, p_k^2, p_k^3)^\top$

Markov propagation: $$\mathbf{p}_k = \mathrm{P}\,\mathbf{p}_{k-1} = \begin{pmatrix} p(1|1) & p(1|2) & p(1|3) \\ p(1|2) & p(1|2) & p(1|2) \\ p(1|3) & p(1|3) & p(1|3) \end{pmatrix} \begin{pmatrix} p_{k-1}^1 \\ p_{k-1}^2 \\ p_{k-1}^3 \end{pmatrix}$$

Perron-Frobenius: the spectral radius of stochastic matrices is 1, 1 is also an eigenvalue and the corresponding eigenvector is positive.

Exercise **Consider the example:** $$\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

and calculate the invariant state (eigenvector for eingenvalue 1). Show numerically or mathematically that each initial state converges to the invariant state.

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Excursus: Stochastic Characterization of Object Interrelations:

Estimation and Tracking of Adjacency Matrices

- Multiple object tracking: estimate from uncertain data Z at each time the kinematic state vector of all relevant objects: $p(x|Z)$.
- Sometimes of interest: interrelations between tracked objects. Example: reachability between two objects (communications, mutual help).
- Interrelations completely described by the adjacency matrix X of a graph (nodes: tracked objects, matrix elements: properties of the interrelation).
- Uncertainty of sensor data (kinematics, attributes) z, Z: adjacency matrix is a random matrix (matrix variate probability densities).
- State to be estimated: kinematics x of all objects, adjacency matrix X. Based on the sensor data, the knowledge on x, X is contained in: $p(x, X|z, Z)$.
- suitable families of matrix variate densities and likelihood functions: Bayes!

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1}) = p(\mathbf{x}_k|j_k, \mathbf{x}_{k-1}, j_{k-1})\, p(j_k|\mathbf{x}_{k-1}, j_{k-1}) \stackrel{!}{=} p(\mathbf{x}_k|\mathbf{x}_{k-1}, j_k)\, p(j_k|j_{k-1}) \quad \text{(Markov)}$$

$$= \underbrace{p(j_k|j_{k-1})}_{\text{interaction}}\ \underbrace{\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1}, j_k)\, p(j_k|j_{k-1}) \quad \text{(Markov)}$$

$$= \underbrace{p(j_k|j_{k-1})}_{\text{interaction}} \; \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

previous posterior written as a Gaussian mixture:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}^{j_{k-1}}_{k-1|k-1},\, \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),$$

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$
\begin{aligned}
p(x_k, j_k | x_{k-1}, j_{k-1}) &= p(x_k | x_{k-1}, j_k)\, p(j_k | j_{k-1}) \\
&= \underbrace{p(j_k | j_{k-1})}_{\text{interaction}} \; \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}
\end{aligned}
$$

previous posterior written as a GAUSSian mixture:

$$
p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}^{j_{k-1}}_{k-1|k-1},\, \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),
$$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r} \sum_{j_{k-1}=1}^{r} \int d\mathbf{x}_{k-1}\; \underbrace{p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1})}_{\text{IMM dynamics}}\; p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1})$$

in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(x_k, j_k|x_{k-1}, j_{k-1}) = p(x_k|x_{k-1}, j_k)\, p(j_k|j_{k-1})$$
$$= \underbrace{p(j_k|j_{k-1})}_{\text{interaction}} \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

previous posterior written as a GAUSSian mixture:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}^{j_{k-1}}_{k-1|k-1},\, \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),$$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r}\sum_{j_{k-1}=1}^{r} \int d\mathbf{x}_{k-1}\, \underbrace{p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1})}_{\text{IMM dynamics}}\, p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1})$$
$$= \sum_{j_k=1}^{r}\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^{j_k j_{k-1}}_{k|k-1},\, \mathbf{P}^{j_k j_{k-1}}_{k|k-1}\big)$$

with $\mathbf{x}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}$, $\mathbf{P}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}\mathbf{F}^{j_k\top}_{k|k+1} + \mathbf{D}^{j_k}_{k|k+1}$ (product formula)

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in practical applications: uncertainty on which dynamics model j_k out of a set of r alternatives is in effect at t_k (IMM: Interacting Multiple Models)

$$p(x_k, j_k | x_{k-1}, j_{k-1}) = p(x_k | x_{k-1}, j_k)\, p(j_k | j_{k-1})$$
$$= \underbrace{p(j_k | j_{k-1})}_{\text{interaction}}\ \underbrace{\mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}^{j_k}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}^{j_k}_{k|k-1}\big)}_{\text{dynamics model } j_k}$$

previous posterior written as a GAUSSian mixture:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1}) = \sum_{j_{k-1}=1}^{r} p(j_{k-1}|\mathcal{Z}^{k-1})\ \mathcal{N}\big(\mathbf{x}_{k-1};\ \mathbf{x}^{j_{k-1}}_{k-1|k-1},\ \mathbf{P}^{j_{k-1}}_{k-1|k-1}\big),$$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{j_k=1}^{r}\sum_{j_{k-1}=1}^{r}\int d\mathbf{x}_{k-1}\ \underbrace{p(\mathbf{x}_k, j_k|\mathbf{x}_{k-1}, j_{k-1})}_{\text{IMM dynamics}}\ p(\mathbf{x}_{k-1}, j_{k-1}|\mathcal{Z}^{k-1})$$
$$= \sum_{j_k=1}^{r}\underbrace{\sum_{j_{k-1}=1}^{r} p(j_k|j_{k-1})\, p(j_{k-1}|\mathcal{Z}^{k-1})\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}^{j_k j_{k-1}}_{k|k-1},\ \mathbf{P}^{j_k j_{k-1}}_{k|k-1}\big)}_{\approx p(j_k|\mathcal{Z}^{k-1})\ \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}^{j_k}_{k|k-1},\, \mathbf{P}^{j_k}_{k|k-1})}$$

alternative: longer model histories!

with $\mathbf{x}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1},\ \mathbf{P}^{j_k j_{k-1}}_{k|k-1} = \mathbf{F}^{j_k}_{k|k+1}\mathbf{x}^{j_{k-1}}_{k-1|k-1}\mathbf{F}^{j_k\top}_{k|k+1} + \mathbf{D}^{j_k}_{k|k+1}$ (product formula)

Approximate GAUSSian mixture representation of $p(\mathbf{x}_k|\mathcal{Z}^{k-1})$ with r mixture components!

BAYESian filtering update based on IMM predictions ($r = 1$: KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k;\, Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k;\, Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{\sum_{j_k=1}^{r} \ell(\mathbf{x}_k;\, Z_k, m_k)\, p(j_k|\mathcal{Z}^{k-1})\ \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k}, \mathbf{P}_{k|k-1}^{j_k}\big)}{\sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k-1})\ \int d\mathbf{x}_k\, \ell(\mathbf{x}_k;\, Z_k, m_k)\ \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k}, \mathbf{P}_{k|k-1}^{j_k}\big)}$$

Bayesian filtering update based on IMM predictions ($r = 1$: KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{\sum_{j_k=1}^{r} \ell(\mathbf{x}_k; Z_k, m_k)\, p(j_k|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k}\big)}{\sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^{k-1})\, \int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k}\big)}$$

Consider as a simple example $\ell(\mathbf{x}_k; Z_k, m_k) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$!

$$= \sum_{j_k=1}^{r} \frac{p(j_k|\mathcal{Z}^{k-1})\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k})}{\sum_{j'_k=1}^{r} p(j'_k|\mathcal{Z}^{k-1})\, \int d\mathbf{x}_k\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j'_k},\, \mathbf{P}_{k|k-1}^{j'_k}\big)}$$

$$= \sum_{j_k=1}^{r} p(j_k|\mathcal{Z}^k)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k}^{j_k},\, \mathbf{P}_{k|k}^{j_k}\big) \qquad \text{(due to the product formula)}$$

Bayesian filtering update based on IMM predictions ($r = 1$: KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k;\, Z_k, m_k)\; p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\; \ell(\mathbf{x}_k;\, Z_k, m_k)\; p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{\sum_{j_k=1}^r \ell(\mathbf{x}_k;\, Z_k, m_k)\; p(j_k|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k}\big)}{\sum_{j_k=1}^r p(j_k|\mathcal{Z}^{k-1})\; \int d\mathbf{x}_k\; \ell(\mathbf{x}_k;\, Z_k, m_k)\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k}\big)}$$

Consider as a simple example $\ell(\mathbf{x}_k;\, Z_k, m_k) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$!

$$= \sum_{j_k=1}^r \frac{p(j_k|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\; \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j_k},\, \mathbf{P}_{k|k-1}^{j_k})}{\sum_{j'_k=1}^r p(j'_k|\mathcal{Z}^{k-1})\; \int d\mathbf{x}_k\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1}^{j'_k},\, \mathbf{P}_{k|k-1}^{j'_k}\big)}$$

$$= \sum_{j_k=1}^r p(j_k|\mathcal{Z}^k)\; \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k}^{j_k},\, \mathbf{P}_{k|k}^{j_k}\big)$$ (due to the product formula)

with: $$p(j_k|\mathcal{Z}^k) = \frac{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_{k|k-1}^{j_k},\, \mathbf{H}_k\mathbf{P}_{k|k-1}^{j_k}\mathbf{H}_k + \mathbf{R}_k\big)}{\sum_{j'_k=1}^r p(j'_k|\mathcal{Z}^{k-1})\; \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_{k|k-1}^{j'_k},\, \mathbf{H}_k\mathbf{P}_{k|k-1}^{j'_k}\mathbf{H}_k + \mathbf{R}_k\big)}$$ (mixture coefficients)

$$\mathbf{x}_{k|k}^{j_k} = \mathbf{x}_{k|k-1}^{j_k} + \mathbf{x}_{k|k}^{j_k}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}^{j_k}), \qquad \mathbf{W}_{k|k}^{j_k} = \mathbf{P}_{k|k-1}^{j_k}\mathbf{H}_k^\top \mathbf{S}_{k|k}^{j_k^{-1}}$$ (KALMAN update)

$$\mathbf{P}_{k|k}^{j_k} = \mathbf{P}_{k|k-1}^{j_k} - \mathbf{W}_{k|k-1}^{j_k}\mathbf{S}_{k|k}^{j_k}\mathbf{W}_{k|k-1}^{j_k}, \qquad \mathbf{S}_{k|k}^{j_k} = \mathbf{H}_k\mathbf{P}_{k|k-1}^{j_k}\mathbf{H}_k + \mathbf{R}_k.$$

Fraunhofer FKIE

IMM Models: Retrodiction

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k)$$ (marginal density!)

IMM Models: Retrodiction

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1},i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k) \qquad \text{(marginal density!)}$$

$$= \sum_{i_{l+1},i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}\mathcal{Z}^k)\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction for } t_{l+1}}$$

for time $l+1$ assume: $p(\mathbf{x}_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1}}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}^k_{i_{l+1}},\, \mathbf{P}^k_{i_{l+1}}\big)$

Fraunhofer FKIE

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$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1},i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k) \qquad \text{(marginal density!)}$$

$$= \sum_{i_{l+1},i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}\mathcal{Z}^k)\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{Retrodiction for } t_{l+1}}$$

for time $l+1$ assume: $p(\mathbf{x}_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1}}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}^k_{i_{l+1}},\, \mathbf{P}^k_{i_{l+1}}\big)$

$$p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^k}) = p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^l}) = \frac{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)\; p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}{\sum_{i_l} \int \mathbf{x}_l\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)}_{\text{IMM model}}\; \underbrace{p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}_{\text{filtering } t_l}}$$

IMM Models: Retrodiction

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1},i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_{l+1}, \mathbf{x}_l, i_{l+1}, i_l|\mathcal{Z}^k) \qquad \text{(marginal density!)}$$

$$= \sum_{i_{l+1},i_l} \int d\mathbf{x}_{l+1}\, p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}\mathcal{Z}^k)\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction for } t_{l+1}}$$

for time $l+1$ assume: $p(\mathbf{x}_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k) = \sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1}}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}^k_{i_{l+1}},\, \mathbf{P}^k_{i_{l+1}}\big)$

$$p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^k}) = p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \boxed{\mathcal{Z}^l}) = \frac{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)\, p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}{\sum_{i_l}\int \mathbf{x}_l\, \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathbf{x}_l, i_l)}_{\text{IMM model}}\, \underbrace{p(\mathbf{x}_l, i_l|\mathcal{Z}^l)}_{\text{filtering } t_l}}$$

$$= \mathbf{c}_{i_{l+1}i_l}(\mathbf{x}_{l+1})\, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}_{i_l} + \mathbf{W}_{i_{l+1}i_l}\big(\mathbf{x}_{l+1} - \mathbf{x}_{i_{l+1}i_l}\big),\, \mathbf{P}_{i_l} - \mathbf{W}_{i_{l+1}i_l}\mathbf{P}_{i_{l+1}i_l}\mathbf{W}_{i_{l+1}i_l}{}^\top\Big) \qquad \textbf{product formula!}$$

with: $\mathbf{c}_{i_{l+1}i_l}(\mathbf{x}_{l+1}) = \dfrac{\boldsymbol{\mu}_{i_{l+1}i_l}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{i_{l+1}i_l},\, \mathbf{P}_{i_{l+1}i_l}\big)}{\sum_{i_l}\boldsymbol{\mu}_{i_{l+1}i_l}\, \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{i_{l+1}i_l},\, \mathbf{P}_{i_{l+1}i_l}\big)}$ $\qquad \mathbf{W}_{i_{l+1}i_l} = \mathbf{P}_{i_l}\mathbf{F}^\top_{i_{l+1}}(\mathbf{F}_{i_{l+1}}\mathbf{P}_{i_l}\mathbf{F}^\top_{i_{l+1}} + \mathbf{D}_{i_{l+1}})^{-1}$

$$\approx \mathbf{c}_{i_{l+1}i_l}(\mathbf{x}^k_{i_{l+1}})$$

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \; \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \; \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

$$= \sum_{i_{l+1}, i_l} \boldsymbol{\mu}^k_{i_{l+1} i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_{l+1} i_l},\, \mathbf{P}^k_{i_{l+1} i_l}\Big)$$

exponential growth of dynamics histories $i_{i_{l+1} i_l \dots}$!

$$p(\mathbf{x}_l|\mathcal{Z}^k) = \sum_{i_{l+1}, i_l} \int d\mathbf{x}_{l+1} \underbrace{p(\mathbf{x}_l, i_l|\mathbf{x}_{l+1}, i_{l+1}, \mathcal{Z}^k)}_{\text{calculated!}} \underbrace{p(\mathbf{x}_{l+1}, i_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction in } t_{l+1}}$$

insert, product formula!

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exponential growth of dynamics histories $i_{i_{l+1} i_l \ldots}$**!**

$$= \sum_{i_l} \underbrace{\sum_{i_{l+1}} \boldsymbol{\mu}^k_{i_{l+1} i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_{l+1} i_l},\, \mathbf{P}^k_{i_{l+1} i_l}\Big)}_{\text{approximation: moment matching!}}$$

finally: $$p(\mathbf{x}_l|\mathcal{Z}^k) \approx \sum_{i_l} \boldsymbol{\mu}^k_{i_l} \, \mathcal{N}\Big(\mathbf{x}_l;\, \mathbf{x}^k_{i_l},\, \mathbf{P}^k_{i_l}\Big)$$

generalize: model histories of *variable* length!

IMM Modeling: Suboptimal Realization

- **Conventional KALMAN filtering**

  Only *one* component: worst-case assumtion

- **standard IMM filter (as discussed!)**

  Approximate after prediction, *before* update by r components! Effort: $\sim r$ KALMAN filter

- **GPB: Generalized Pseudo-BAYESian**

  Approximate *after* measurement processing by r components! Effort: $\sim r^2$ KALMAN filter

- **IMM-MHT filter (nearly optimal)**

  Accept longer dynamics histories $\rightarrow$ *variable* number of components!

Extendable to ambiguity with respect to sensor models!

Fraunhofer FKIE

Improved Approximation: Simple Approach

Consider *dynamics histories* of length κ: $\mathbf{i}_k = (i_k, i_{k-1}, \ldots, i_{k-\kappa+1})$

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als Filterung: $$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \underbrace{\sum_{\mathbf{i}_{k-1}}}_{\kappa \text{ Summen}} \boldsymbol{\mu}_{\mathbf{i}_{k-1}} \, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{\mathbf{i}_{k-1}},\, \mathbf{P}_{\mathbf{i}_{k-1}}\big)$$

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prediction:

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{i_k, \mathbf{i}_{k-1}} \int d\mathbf{x}_{k-1} \, p(\mathbf{x}_k, i_k, \mathbf{x}_{k-1}, \mathbf{i}_{k-1}|\mathcal{Z}^{k-1})$$

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Prďż˝diktion:

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{i_k, \mathbf{i}_{k-1}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k, \mathbf{x}_{k-1}, \mathbf{i}_{k-1}|\mathcal{Z}^{k-1})$$

$$= \underbrace{\sum_{i_k, \ldots, i_{k-\kappa}}}_{\kappa+1 \text{ Summen}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k|\mathbf{x}_{k-1}, i_{k-1})\, p(\mathbf{x}_{k-1}, i_{k-1}, \ldots, i_{k-\kappa}|\mathcal{Z}^{k-1})$$

Improved Approximation: Simple Approach

Consider *dynamics histories* of length κ: $\quad \mathbf{i}_k = (i_k, i_{k-1}, \dots, i_{k-\kappa+1})$

as filtering:

$$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \underbrace{\sum_{\mathbf{i}_{k-1}}}_{\kappa \text{ sums}} \boldsymbol{\mu}_{\mathbf{i}_{k-1}} \, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{\mathbf{i}_{k-1}},\, \mathbf{P}_{\mathbf{i}_{k-1}}\big)$$

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$$= \underbrace{\sum_{i_k,\dots,i_{k-\kappa}}}_{\kappa+1 \text{ sums}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k|\mathbf{x}_{k-1}, i_{k-1})\, p(\mathbf{x}_{k-1}, i_{k-1}, \dots, i_{k-\kappa}|\mathcal{Z}^{k-1})$$

$$= \sum_{i_k,\dots,i_{k-\kappa}} \boldsymbol{\mu}_{i_k,\dots,i_{k-\kappa}}\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{i_k,\dots,i_{k-\kappa}},\, \mathbf{P}_{i_k,\dots,i_{k-\kappa}}\big) = \sum_{\mathbf{i}_k} \underbrace{\sum_{i_{k-\kappa}} \boldsymbol{\mu}_{\mathbf{i}_k, i_{k-\kappa}}\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{\mathbf{i}_k, i_{k-\kappa}},\, \mathbf{P}_{\mathbf{i}_k, i_{k-\kappa}}\big)}_{\text{second order approximation!}}$$

Fraunhofer FKIE

Improved Approximation: Simple Approach

Consider *dynamics histories* of length κ: $\quad \mathbf{i}_k = (i_k, i_{k-1}, \dots, i_{k-\kappa+1})$

as filtering: $$p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \underbrace{\sum_{\mathbf{i}_{k-1}}}_{\kappa \text{ sums}} \boldsymbol{\mu}_{\mathbf{i}_{k-1}} \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{\mathbf{i}_{k-1}},\, \mathbf{P}_{\mathbf{i}_{k-1}}\big)$$

prediction: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{i_k, \mathbf{i}_{k-1}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k, \mathbf{x}_{k-1}, \mathbf{i}_{k-1}|\mathcal{Z}^{k-1})$$

$$= \underbrace{\sum_{i_k, \dots, i_{k-\kappa}}}_{\kappa+1 \text{ sums}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k|\mathbf{x}_{k-1}, i_{k-1})\, p(\mathbf{x}_{k-1}, i_{k-1}, \dots, i_{k-\kappa}|\mathcal{Z}^{k-1})$$

$$= \sum_{i_k, \dots, i_{k-\kappa}} \boldsymbol{\mu}_{i_k, \dots, i_{k-\kappa}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{i_k, \dots, i_{k-\kappa}},\, \mathbf{P}_{i_k, \dots, i_{k-\kappa}}\big) = \sum_{\mathbf{i}_k} \underbrace{\sum_{i_{k-\kappa}} \boldsymbol{\mu}_{\mathbf{i}_k, i_{k-\kappa}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{\mathbf{i}_k, i_{k-\kappa}},\, \mathbf{P}_{\mathbf{i}_k, i_{k-\kappa}}\big)}_{\text{second order approximation!}}$$

finally: $$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \approx \sum_{\mathbf{i}_k} \boldsymbol{\mu}^*_{\mathbf{i}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}^*_{\mathbf{i}_k},\, \mathbf{P}^*_{\mathbf{i}_{k-1}}\big)$$ (for retrodiction analogous!)

Generalization to Ambiguous Sensor Data:

Calculate the pdfs $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \sum_{H_{k-1}} p(\mathbf{x}_k, H_{k-1} \mid \mathcal{Z}^{k-1})$!

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$$p(\mathbf{x}_k, H_{k-1}|\mathcal{Z}^{k-1}) = \sum_{i_k, i_{k-1}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k, \mathbf{x}_{k-1}, i_{k-1}, H_{k-1}|\mathcal{Z}^{k-1})$$

Generalization to Ambiguous Sensor Data:

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$$p(\mathbf{x}_k, H_{k-1}|\mathcal{Z}^{k-1}) = \sum_{i_k, i_{k-1}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k, \mathbf{x}_{k-1}, i_{k-1}, H_{k-1}|\mathcal{Z}^{k-1})$$

$$= \sum_{i_k, i_{k-1}} \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, i_k, \mathbf{x}_{k-1}, i_{k-1}| \underbrace{H_{k-1}, \mathcal{Z}^{k-1})}_{\text{unique!}} \underbrace{p(H_{k-1}|\mathcal{Z}^{k-1})}_{\text{weight: filtering}}$$

calculation: as before!

Design of IMM Modelling

- ***number*** r ***of models:*** **relevant only for standard IMM**
- ***decisive:*** **sufficiently many Gausßian picture components**
- ***irrelevant:*** **by** r **or length of dynamics khistories** n_H
- ***recommendation:*** **worst/best case, histories (**$r = 2$**,** $n_H = 3$**)**
- ***benefit:*** **interpretable, close-to-reality dynamics parameters**

Demonstration