Recapitulation: Description of the Detection Process

Detector: receives signals and decides on object existence  
Processor: processes detected signals and produces measurements

‘\(D\)’: detector detects an object  
\(D\): object actually existent
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\[ D \]: object actually existent

error of 1. kind: \[ P_I = P(\neg 'D' | D) \]

error of 2. kind: \[ P_{II} = P('D' | \neg D) \]

measure of detection performance: \[ P_D = P('D' | D) \]

detector properties characterized by two parameters:

- detection probability \[ P_D = 1 - P_I \]
- false alarm probability \[ P_F = P_{II} \]
Recapitulation: Description of the Detection Process

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detector properties characterized by two parameters:

- detection probability \(P_D = 1 - P_I\)
- false alarm probability \(P_F = P_{II}\)

example (Swerling I model): \(P_D = P_D(P_F, \text{SNR}) = P_F^{1/(1+\text{SNR})}\)

**detector design:** Maximize detection probability \(P_D\) for a given, predefined false alarm probability \(P_F\)!
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

- \(n_k + 1\) possible interpretations of the sensor data \(Z_k = \{z^j_k\}_{j=1}^{n_k}\):
  - \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
  - \(E_j, j = 1, \ldots, n_k\): Object detected; \(z^j_k\) is object measurement; \(n_k - 1\) false measurements

Consider the interpretations in the likelihood function \(p(Z_k, n_k | x_k)\)!
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\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k) \quad D = \text{“object was detected”}
\]
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\]

\[
= p(Z_k, n_k|\neg D, x_k) \frac{P(\neg D|x_k)}{=1-P_D} + p(Z_k, n_k|D, x_k) \frac{P(D|x_k)}{=P_D}
\]

sensor parameter: detection probability \(P_D\)
ambiguuous sensor data \((P_D < 1, \rho_F > 0)\)

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\]

false measurements: Poisson distributed in #, uniformly distributed in the FoV
Modeling of False Measurements (FM)

- Probability of having \( n \) FM: \( p_F(n) \)
  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    \[ \bar{n} = \rho_F |\text{FoV}|, \quad \text{false measurement density } \rho_F \text{ (perhaps not constant)} \]
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    \[ p_F(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}} \]
    expectation: \( \mathbb{E}[n] = \bar{n} \), variance: \( \mathbb{V}[n] = \bar{n} \)
normalization: \[ \sum_{n=0}^{\infty} p_F(n) = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = e^{-\bar{n}} e^{\bar{n}} = 1 \]

expectation: \[ \mathbb{E}[n] = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = e^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^n}{n!} = \bar{n} e^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^{n-1}}{(n-1)!} = \bar{n} \]

variance: \[ \nabla[n] = \mathbb{E}[(n - \bar{n})^2] = \mathbb{E}[n^2] - \bar{n}^2 = \ldots \text{exercise!} \ldots = \bar{n} \]
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- **Distribution of FM in the Field of View:** $p(z_1^f, \ldots, z_n^f|\text{FoV})$
  - FM mutually independent:
    \[ p(z_1^f, \ldots, z_n^f|\text{FoV}) = \prod_{i=1}^{n} p(z_i^f|\text{FoV}) \]
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  - uniformly distributed in the FoV:
    \[ p(z_i^f | \text{FoV}) = \frac{1}{|\text{FoV}|} \quad \text{(often!)} \]
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

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\[
= p(Z_k|n_k, \neg D, x_k) p(n_k|\neg D, x_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, x_k)
\]

\[
= |\text{FoV}|^{-n_k} p_F(n_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} \frac{p(Z_k|n_k, j, D, x_k) p(j|n_k, D) p(n_k|D)}{|\text{FoV}|^{-n_k-1} N(z^j_k; Hx_k, R)} = 1/n_k \Rightarrow p_F(n_k - 1)
\]

Insert Poisson distribution: \(p_F(n_k) = \frac{(\rho_F|\text{FoV}|^{-n_k})}{n_k!} e^{-\rho_F|\text{FoV}|}\)
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)


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\[
= |\text{FoV}|^{-n_k} p_F(n_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k|n_k, j, D, x_k) p(j|n_k, D) p(n_k|D)
\]

\[
= e^{-\rho_F|\text{FoV}|/n_k!} \rho_F^{n_k} \left((1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}(z^j_k; \mathbf{Hx}_k, \mathbf{R})\right)
\]
Likelihood Functions

The likelihood function answers the question:

What does the sensor tell about the state \( x \) of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object**: \( P_D = 1, \rho_F = 0 \)

  at each time one measurement:

  \[
  p(z_k| x_k) = \mathcal{N}(z_k; Hx_k, R)
  \]

- **real conditions, one object**: \( P_D < 1, \rho_F > 0 \)

  at each time \( n_k \) measurements \( Z_k = \{z_k^1, \ldots, z_k^{n_k}\} \):

  \[
  p(Z_k, n_k| x_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}(z_k^j; Hx_k, R)
  \]
Bayes Filtering for: $P_D < 1$, $\rho_F > 0$, well-separated objects

state $x_k$, current data $Z_k = \{z_k^j\}_{j=1}^{m_k}$, accumulated data $\mathcal{Z}^k = \{Z_k, \mathcal{Z}^{k-1}\}$

interpretation hypotheses $E_k$ for $Z_k$
object not detected, $1 - P_D$
$z_k \in Z_k$ from object, $P_D$
$m_k + 1$ interpretations

interpretation histories $H_k$ for $\mathcal{Z}^k$
• tree structure: $H_k = (E_{H_k}, H_{k-1}) \in \mathcal{H}^k$
• current: $E_{H_k}$, prehistories: $H_{k-i}$

$$p(x_k | \mathcal{Z}^k) = \sum_{H_k} p(x_k, H_k | \mathcal{Z}^k) = \sum_{H_k} \underbrace{p(H_k | \mathcal{Z}^k)}_{\text{weight!}} \underbrace{p(x_k | H_k, \mathcal{Z}^k)}_{\text{given } H_k: \text{ unique}}$$

‘mixture’ density
Closer look: $P_D < 1$, $\rho_F > 0$, well-separated targets

filtering (at time $t_{k-1}$):

$$p(x_{k-1}|Z^{k-1}) = \sum_{H_{k-1}} p_{H_{k-1}} \mathcal{N}(x_{k-1}; x_{H_{k-1}}, P_{H_{k-1}})$$

prediction (for time $t_k$):

$$p(x_k|Z^{k-1}) = \int dx_{k-1} p(x_k|x_{k-1}) p(x_{k-1}|Z^{k-1}) \quad \text{(MARKOV model)}$$

$$= \sum_{H_{k-1}} p_{H_{k-1}} \mathcal{N}(x_k; Fx_{H_{k-1}}, FP_{H_{k-1}}F^T + D) \quad \text{(IMM also possible)}$$

measurement likelihood:

$$p(Z_k, m_k|x_k) = \sum_{j=0}^{m_k} p(Z_k|E^j_k, x_k, m_k) P(E^j_k|x_k, m_k) \quad \text{($E^j_k$: interpretations)}$$

$$\propto (1 - P_D) \rho_F + P_D \sum_{j=1}^{m_k} \mathcal{N}(z^j_k; Hx_k, R) \quad \text{($H, R, P_D, \rho_F$)}$$

filtering (at time $t_k$):

$$p(x_k|Z^k) \propto p(Z_k, m_k|x_k) p(x_k|Z^{k-1}) \quad \text{(BAYES' rule)}$$

$$= \sum_{H_k} p_{H_k} \mathcal{N}(x_k; x_{H_k}, P_{H_k}) \quad \text{(Exploit product formula)}$$
Problem: Growing Memory Disaster:

\[ m \text{ data, } N \text{ hypotheses } \rightarrow N^{m+1} \text{ continuations} \]

radical solution: mono-hypothesis approximation
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- **gating**: Exclude competing data with \[ ||X^i_{k|k-1}|| > \lambda! \]

\[ \rightarrow \text{KALMAN filter (KF)} \]

+ very simple, – \( \lambda \) too small: loss of target measurement
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- Force a **unique interpretation** in case of a conflict!

  look for *smallest statistical distance*: \(\min_i |\nu^i_{k|k-1}|\)

  - Nearest-Neighbor filter (NN)
Problem: Growing Memory Disaster:

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- Force a **unique interpretation** in case of a conflict!

  look for *smallest statistical distance*: \( \min_i \| \nu_{k|k-1}^i \| \)

  \[ \text{Nearest-Neighbor filter (NN)} \]

  + one hypothesis, – hard decision, – not adaptive

- **global combining**: Merge all hypotheses!

  \[ \text{PDAF, JPDAF filter} \]

  + all data, + adaptive, – reduced applicability
Prädiktion

Gates

Messungen

Prädiktion

PDAF

NN
Prädiktion

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NN

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MHT
\textbf{Moment Matching:} Approximate an arbitrary pdf \( p(x) \) with \( \mathbb{E}[x] = x, \mathbb{C}[x] = P \) by \( p(x) \approx \mathcal{N}(x; x, P) \)!

here especially: \( p(x) = \sum_H p_H \mathcal{N}(x; x_H, P_H) \) (normal mixtures)

\[
\begin{align*}
x &= \sum_H p_H x_H \\
P &= \sum_H p_H \left\{ P_H + (x_H - x)(x_H - x)^T \right\}
\end{align*}
\]
PDAF Filter: formally analogous to Kalman Filter

Filtering (scan $k-1$): 
\[ p(x_{k-1}|Z^{k-1}) = \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \] (→ initiation)

Prediction (scan $k$): 
\[ p(x_k|Z^{k-1}) \approx \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \] (like Kalman)

Filtering (scan $k$): 
\[ p(x_k|Z^k) \approx \sum_{j=0}^{m_k} p_k^j \mathcal{N}(x_k; x_{k|k}, P_{k|k}) \approx \mathcal{N}(x_k; x_k, P_{k|k}) \]
PDAF Filter: formally analogous to Kalman Filter

Filtering (scan \( k - 1 \)): \[
p(x_{k-1}|Z^{k-1}) = \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \quad (\text{→ initiation})
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prediction (scan \( k \)): \[
p(x_k|Z^{k-1}) \approx \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \quad (\text{like Kalman})
\]

Filtering (scan \( k \)): \[
p(x_k|Z^k) \approx \sum_{j=0}^{m_k} p_j^k \mathcal{N}(x_k; x_{j|k}, P_{j|k})
\]

\[
x_{j|k} = \begin{cases} x_{k|k-1} \\
     x_{k|k-1} + W_k
    \end{cases}
\quad
p_j^k = \begin{cases} P_{j|k-1} \\
     P_{j|k-1} - W_k S_k W_k^\top
    \end{cases}
\]

\[
\nu_j = z_j - H x_k,
\quad
W_k = \frac{P_{k|k-1} H^\top S_k^{-1}}{\text{gain matrix}}
\]
\[
S_k = H P_{k|k-1} H^\top + R_k
\]

\[
p_k^* = \left( \begin{array}{c}
    (1 - P_D) \\
    P_D \\
  \end{array} \right) \rho_F
d\quad
p_j^* = \left( \begin{array}{c}
    \frac{1}{\sqrt{2\pi S_{j|k}}} \\
    e^{-\frac{1}{2} \nu_{j|k}^\top S_{j|k} \nu_{j|k}}
  \end{array} \right)
\]
Second-order Approximation of the Mixture Density:

\[
\sum_{j=1}^{m_k} p_j^k \mathcal{N}(x_k; x_{k|k}^j, P_{k|k}^j) \approx \mathcal{N}(x_k; x_{k|k}, P_{k|k})
\]

mit:

\[
x_{k|k} = \sum_{j=0}^{m_k} p_k^j x_{k|k}^j
\]

\[
P_{k|k} = \sum_{j=0}^{m_k} p_k^j \left( P_{k|k}^j + (x_{k|k}^j - x_{k|k})(x_{k|k}^j - x_{k|k})^T \right)
\]
\[ x_{k|k} = \sum_{j=0}^{m_k} p_k^j x_{k|k}^j, \quad x_{k|k}^0 = x_{k|k-1}, \quad x_{k|k}^j = x_{k|k-1} + W_k \nu_k^j \]

\[ P_{k|k} = \sum_{j=0}^{m_k} p_k^j (P_{k|k}^j + (x_{k|k}^j - x_{k|k}) (x_{k|k}^j - x_{k|k})^\top) \]
\[
\begin{align*}
\mathbf{x}_{k|k} &= \sum_{j=0}^{m_k} p_k^j \mathbf{x}_{k|k}^j \\
&= p_k^0 \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_k^j \left( \mathbf{x}_{k|k-1} + \mathbf{W}_k \mathbf{v}_k^j \right) \\
\mathbf{P}_{k|k} &= \sum_{j=0}^{m_k} p_k^j \left( \mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top \right)
\end{align*}
\]
\[ x_{k|k} = \sum_{j=0}^{m_k} p^j_k x^j_{k|k} \]

\[ = p^0_k x_{k|k-1} + \sum_{j=1}^{m_k} p^j_k (x_{k|k-1} + W_k \nu^j_k) \]

\[ = x_{k|k-1} \left( p^0_k + \sum_{j=1}^{m_k} p^j_k \right) + W_k \sum_{j=1}^{m_k} p^j_k \nu^j_k \]

\[ \text{mean!} \]

\[ P_{k|k} = \sum_{j=0}^{m_k} p^j_k (P^j_{k|k} + (x^j_{k|k} - x_{k|k})(x^j_{k|k} - x_{k|k})^\top) \]
\[
\mathbf{x}_{k|k} = \sum_{j=0}^{m_k} p_j^k \mathbf{x}_{k|k}^j
\]
\[
= p_0^k \mathbf{x}_{k|k-1} + \sum_{j=1}^{m_k} p_j^k (\mathbf{x}_{k|k-1} + \mathbf{W}_k \mathbf{v}_k^j) = \mathbf{x}_{k|k-1} + \mathbf{W}_k \mathbf{v}_k
\]
\[
\mathbf{P}_{k|k} = \sum_{j=0}^{m_k} p_j^k (\mathbf{P}_{k|k}^j + (\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})(\mathbf{x}_{k|k}^j - \mathbf{x}_{k|k})^\top)
\]

**Combined Innovation:**
\[
\mathbf{v}_k = \sum_{j=1}^{m_k} p_j^k \mathbf{v}_k^j
\]
\[ x_{k|k} = \sum_{j=0}^{m_k} p^j_k x_{k|k}^j \]

\[ = p^0_k x_{k|k-1} + \sum_{j=1}^{m_k} p^j_k (x_{k|k-1} + W_k \nu_k^j) = x_{k|k-1} + W_k \nu_k \]

\[ P_{k|k} = \sum_{j=0}^{m_k} p^j_k (P_{k|k}^j + (x_{k|k}^j - x_{k|k}) (x_{k|k}^j - x_{k|k})^\top) \]

\[ = P_{k|k-1} - \sum_{j=1}^{m_k} p^j_k W_k S_k W_k^\top + \sum_{j=1}^{m_k} p^j_k W_k (\nu_k^j - \nu_k) (\nu_k^j - \nu_k)^\top W_k^\top \]

**Combined Innovation:** \[ \nu_k = \sum_{j=1}^{m_k} p^j_k \nu_k^j \]
\[ x_{k|k} = \sum_{j=0}^{m_k} p_k^j x_{k|k}^j \]
\[ = p_k^0 x_{k|k-1} + \sum_{j=1}^{m_k} p_k^j (x_{k|k-1} + W_k \nu_k^j) = x_{k|k-1} + W_k \nu_k \]

\[ P_{k|k} = \sum_{j=0}^{m_k} p_k^j (P_k^j + (x_{k|k}^j - x_{k|k})(x_{k|k}^j - x_{k|k})^\top) \]
\[ = P_{k|k-1} - \sum_{j=1}^{m_k} p_k^j W_k S_k W_k^\top + \sum_{j=1}^{m_k} p_k^j W_k (\nu_k^j - \nu_k) (\nu_k^j - \nu_k)^\top W_k^\top \]
\[ = P_{k|k-1} - (1 - p_k^0) W_k S_k W_k^\top + W_k \left[ \sum_{j=1}^{m_k} p_k^j \nu_k^j \nu_k^j - \nu_k \nu_k^\top \right] W_k^\top \]

**Combined Innovation:** \[ \nu_k = \sum_{j=1}^{m_k} p_k^j \nu_k^j \]
PDAF Filter: formally analog to Kalman Filter

Filtering (scan $k-1$):  \[ p(x_{k-1}|Z^{k-1}) = \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \] (→ initiation)

Prediction (scan $k$):  \[ p(x_k|Z^{k-1}) \approx \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \] (like Kalman)

Filtering (scan $k$):  \[ p(x_k|Z^{k}) \approx \sum_{j=0}^{m_k} p^j_k \mathcal{N}(x_k; x_{k|k}, P_{k|k}) = \mathcal{N}(x_k; x_{k|k}, P_{k|k}) \]

\[ \nu_k = \sum_{j=0}^{m_k} p^j_k \nu_k^j, \quad \nu_k^j = z_k^j - Hx_{k|k-1} \] combined innovation

\[ W_k = P_{k|k-1}H^TS_k^{-1}, \quad S_k = HP_{k|k-1}H^T + R_k \] Kalman gain matrix

\[ p^j_k = p^j_{k*} / \sum_j p^j_{k*}, \quad p^j_{k*} = \left\{ \frac{(1-P_D)p^j_F}{\sqrt{2\pi S_k}} \right\} e^{-\frac{1}{2}S_k^T S_k \nu_k^j} \] weighting factors

\[ x_k = x_{k|k-1} + W_k \nu_k \] (Filtering Update: Kalman)

\[ P_k = P_{k|k-1} - (1-p^0_k) W_k S_k W_k^T \] (Kalman part)

\[ + W_k \left\{ \sum_{j=0}^{m_k} p^j_k \nu_k^j \nu_k^j - \nu_k \nu_k^T \right\} W_k^T \] (Spread of Innovations)
PDAF: Characteristic Properties

- filtering: processing of *combined innovation*
- *all data* $Z_k$ in the gate are considered
- $p_i$ data dependent! Update *not linear*
- missing measurement: $P_{k|k-1}$ with weight $p_0$
- "usual" Kalman covariance according to $(1 - p_0)$
- Spread *positively semidefinite*: larger covariance
- therefore: *data driven adaptivity*
- *non linear estimator*: data dependent error
- Performance prediction *only via simulations*

**Problem:** Multimodalität geht verloren!
The qualitative shape of $p(x_k|z^k)$ is often much simpler than its correct representation: *a few pronounced modes*

**adaptive solution:** nearly optimal approximation
The qualitative shape of \( p(x_k | Z^k) \) is often much simpler than its correct representation: a few pronounced modes

**adaptive solution: nearly optimal approximation**

- **individual gating:** Exclude *irrelevant data*!
  before continuing existing track hypotheses \( H_{k-1} \)
  → *limiting case*: KALMAN filter (KF)
The qualitative shape of $p(x_k|Z^k)$ is often much simpler than its correct representation: \textit{a few pronounced modes}

**adaptive solution: nearly optimal approximation**

- **individual gating:** Exclude \textit{irrelevant data!}
  before continuing existing track hypotheses $H_{k-1}$
  $\rightarrow$ \textit{limiting case: KALMAN filter (KF)}

- **pruning:** Kill hypotheses of very \textit{small weight!}
  after calculating the weights $p_{H_k}$, before filtering
  $\rightarrow$ \textit{limiting case: Nearest Neighbor filter (NN)}
The qualitative shape of $p(x_k | Z^k)$ is often much simpler than its correct representation: *a few pronounced modes*

**Adaptive solution: Nearly optimal approximation**

- **Individual gating**: Exclude *irrelevant data*!
  
  before continuing existing track hypotheses $H_{k-1}$
  
  → *limiting case*: Kalman filter (KF)

- **Pruning**: Kill hypotheses of very *small weight*!
  
  after calculating the weights $p_{H_k}$, before filtering
  
  → *limiting case*: Nearest Neighbor filter (NN)

- **Local combining**: Merge *similar hypotheses*!
  
  after the complete calculation of the pdfs
  
  → *limiting case*: PDAF (global combining)
**Successive Local Combining**

Partial sums of *similar* densities $\rightarrow$ moment matching:

$$
\sum_{H_k \in \mathcal{H}_k^*} p_{H_k} \mathcal{N}(x_k; x_{H_k}, P_{H_k}) \approx p_{H_k^*} \mathcal{N}(x_k; x_{H_k^*}, P_{H_k^*})
$$

$\mathcal{H}_k^* \subset \mathcal{H}_k$ $\rightarrow$ $H_k^*$: *effective* hypothesis
Successive Local Combining

Partial sums of similar densities $\rightarrow$ moment matching:

$$\sum_{H_k \in \mathcal{H}_k^{*}} p_{H_k} \mathcal{N}(x_k; x_{H_k}, P_{H_k}) \approx p_{H_k^{*}} \mathcal{N}(x_k; x_{H_k^{*}}, P_{H_k^{*}})$$

$\mathcal{H}_k^{*} \subset \mathcal{H}_k$ $\rightarrow$ $H_k^{*}$: effective hypothesis

similarity: $d(H_1, H_2) < \mu$ mit (z.B.):

$$d(H_1, H_2) = (x_{H_1} - x_{H_2})^\top (P_{H_1} + P_{H_2})^{-1} (x_{H_1} - x_{H_2})$$

**Start:** Hypothesis of highest weight $H_1$ $\rightarrow$ search similar hypothesis $(p_H \backslash \alpha)$ $\rightarrow$ merge: $(H_1, H) \succ H_1^{*} \rightarrow$ continue search $(p_H \backslash \alpha)$ ...

$\rightarrow$ **restart:** hypothesis with next to highest weight $H_2$ $\rightarrow$ ...
Successive Local Combining

Partial sums of similar densities $\rightarrow$ moment matching:

$$\sum_{H_k \in \mathcal{H}^{k*}} p_{H_k} \mathcal{N}(x_k; x_{H_k}, P_{H_k}) \approx p_{H_k^*} \mathcal{N}(x_k; x_{H_k^*}, P_{H_k^*})$$

$\mathcal{H}^{k*} \subset \mathcal{H}^k$ $\rightarrow$ $H_k^*$: effective hypothesis

**similarity:** $d(H_1, H_2) < \mu$ mit (z.B.):

$$d(H_1, H_2) = (x_{H_1} - x_{H_2})^\top (P_{H_1} + P_{H_2})^{-1} (x_{H_1} - x_{H_2})$$

**Start:** Hypothesis of highest weight $H_1$ $\rightarrow$ search similar hypothesis $(p_H \downarrow_\lambda) \rightarrow$ merge: $(H_1, H) > H_1^* \rightarrow$ continue search $(p_H \downarrow_\lambda) \ldots$ $\rightarrow$ **restart:** hypothesis with next to highest weight $H_2$ $\rightarrow$ ...

- In many cases: good approximations $\rightarrow$ quasi-optimality
- PDAF, JPDAF: $\mathcal{H}^{k*} = \mathcal{H}^k$ $\rightarrow$ limited applicability
- robustness $\rightarrow$ detail mostly irrelevant