Kalman filter: general properties

$$
\begin{aligned}
\mathbf{x}_{k|k} &= \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\, \boldsymbol{\nu}_{k|k-1} \\
\mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}^{\top}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}_{k|k-1} &= \mathbf{H}_{k|k-1}\mathbf{P}_{k|k-1}\mathbf{H}_{k|k-1}^{\top} + \mathbf{R}_{k|k-1} \\
\mathbf{W}_{k|k-1} &= \mathbf{P}_{k|k-1}\mathbf{H}_{k|k-1}^{\top}\mathbf{S}_{k|k-1}^{-1}
\end{aligned}
$$

- *all elements* of the density: estimate an quality measure
- *variable* update, *time*-dependent evolution, measurement error
- variable *type*: measurement matrix (e.g. incomplete measurements)
- low computational effort (e.g. analytic inversions)

A (very!) useful product formula for GAUSSians

$$\mathcal{N}(\mathbf{z};\ \mathbf{H}\mathbf{x},\ \mathbf{R})\,\mathcal{N}(\mathbf{x};\ \mathbf{y},\ \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\ \mathbf{H}\mathbf{y},\ \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\ \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\ \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\ \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\ \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Sketch of the proof:

- Interpret $\mathcal{N}(\mathbf{z};\ \mathbf{H}\mathbf{x},\ \mathbf{R})\,\mathcal{N}(\mathbf{x};\ \mathbf{y},\ \mathbf{P})$ as a joint pdf $p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{z},\mathbf{x})$.
- Show that $p(\mathbf{z},\mathbf{x})$ is a GAUSSian: $p(\mathbf{z},\mathbf{x}) = \mathcal{N}\left(\binom{\mathbf{z}}{\mathbf{x}};\ \binom{\mathbf{H}\mathbf{y}}{\mathbf{y}},\ \begin{pmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{pmatrix}\right)$.
- Calculate from $p(\mathbf{z},\mathbf{x})$ the marginal and conditional pdfs $p(\mathbf{z})$ and $p(\mathbf{x}|\mathbf{z})$.
- From $p(\mathbf{z},\mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) = p(\mathbf{x},\mathbf{z})$ we obtain the result.

Filtering step: alternative formulation

$$p(\mathbf{x}_k | \mathcal{Z}^k) = p(\mathbf{x}_k | \mathbf{z}_k, \mathcal{Z}^{k-1}) \qquad \text{(current measurement)}$$

$$= \frac{p(\mathbf{z}_k | \mathbf{x}_k)\, p(\mathbf{x}_k | \mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, p(\mathbf{z}_k | \mathbf{x}_k)\, p(\mathbf{x}_k | \mathcal{Z}^{k-1})} \qquad \text{(Bayes' rule)}$$

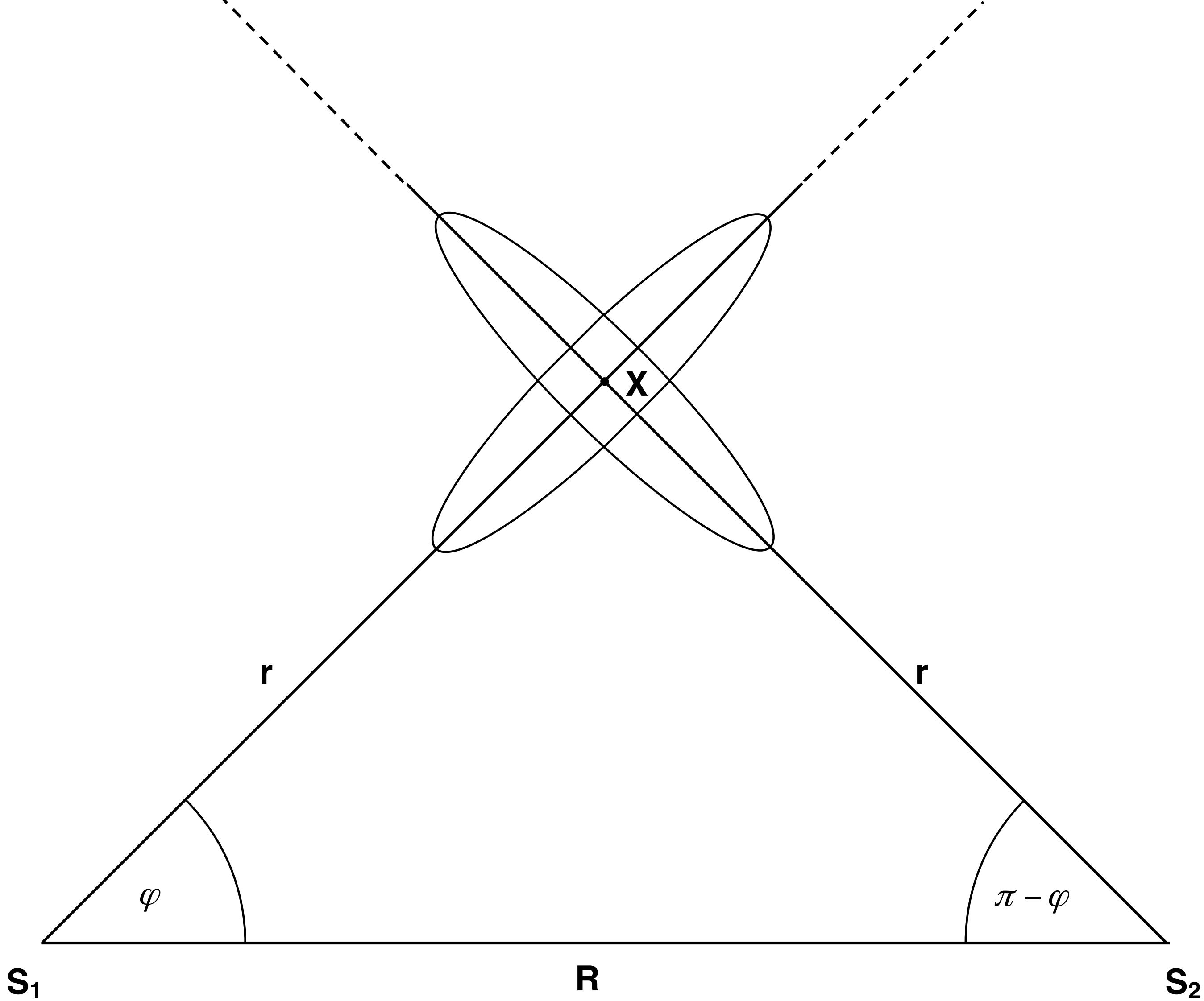
$$= \frac{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)\; \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}{\int d\mathbf{x}_k\, \underbrace{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)}_{\text{likelihood function}}\; \underbrace{\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}_{\text{prediction for } t_k}}$$

$$= \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}) \qquad \text{(product formula: 2. version!)}$$

$$\mathbf{x}_{k|k} = \mathbf{P}_{k|k}^{-1}\left(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{z}_k\right)$$

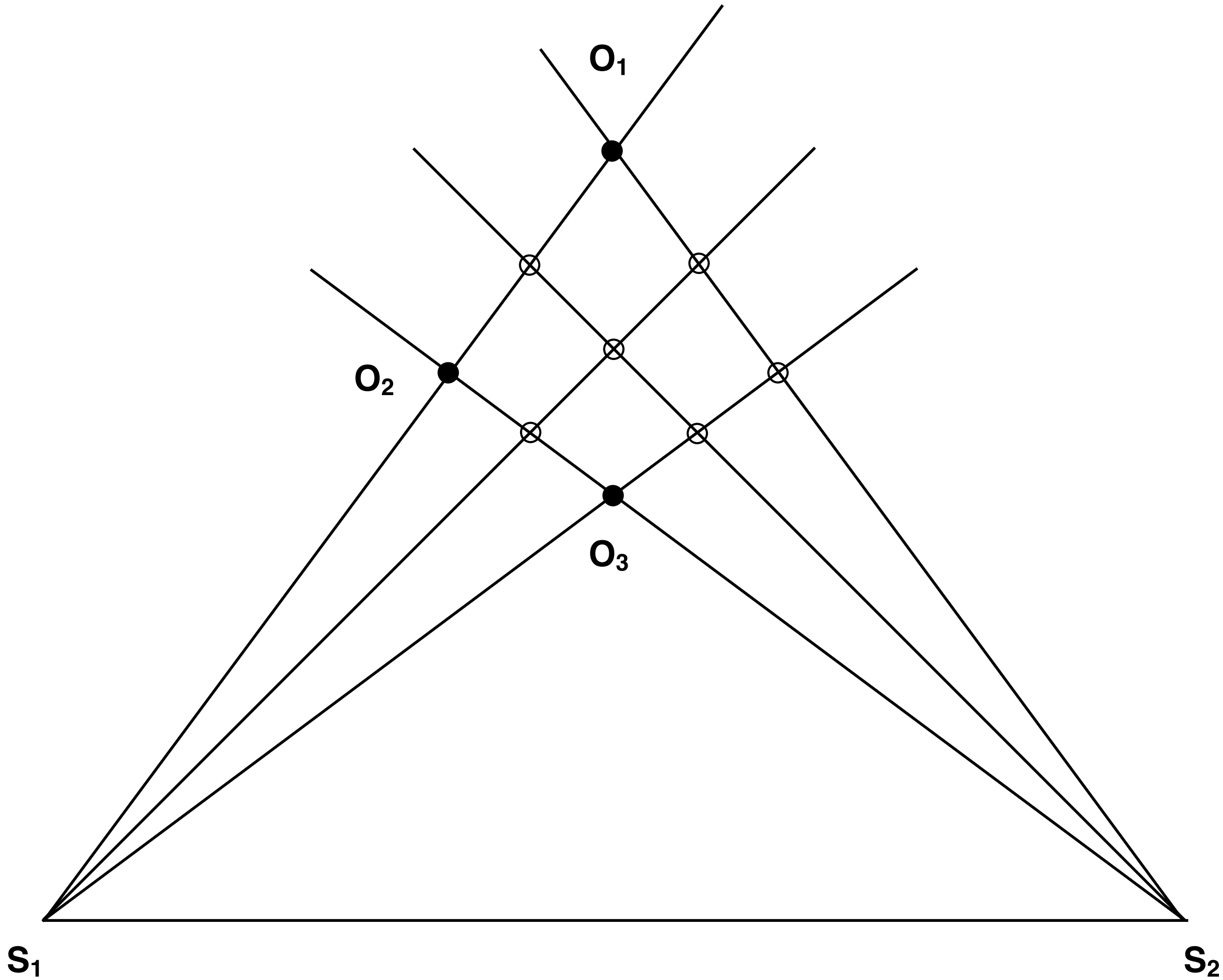
$$\mathbf{P}_{k|k}^{-1} = \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{H}$$

inverse covariance matrices are called information matrices.



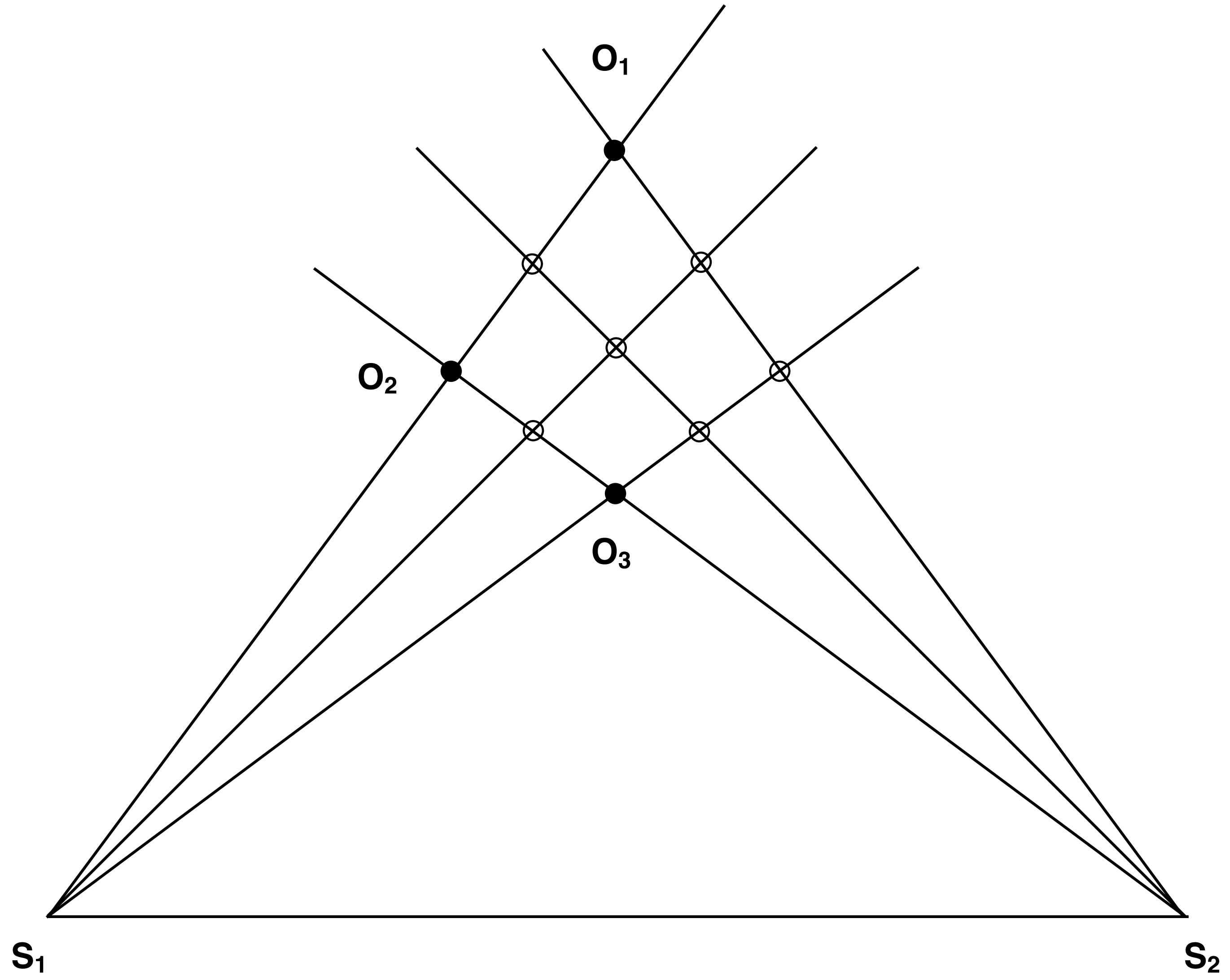
X
r
r
φ
$\pi - \varphi$
S_1
R
S_2

Multiple Objects with Angle-only measurements: Ghosts



Fraunhofer
FKIE

Multiple Objects with Angle-only measurements: Ghosts



Exercise 5.2

Why is the picture wrong?

A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

$$p(\mathbf{z}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\; p(\mathbf{z}_k, \mathbf{x}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\; p(\mathbf{z}_k|\,\mathbf{x}_k)\; p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})$$

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A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

$$p(\mathbf{z}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\; p(\mathbf{z}_k, \mathbf{x}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\; p(\mathbf{z}_k|\,\mathbf{x}_k)\; p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})$$

$$= \int d\mathbf{x}_k\; \underbrace{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)}_{\text{likelihood: sensor model}}\; \underbrace{\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}_{\text{prediction at time } t_k}$$

A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

$$p(\mathbf{z}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\, p(\mathbf{z}_k, \mathbf{x}_k|\,\mathcal{Z}^{k-1}) = \int d\mathbf{x}_k\, p(\mathbf{z}_k|\,\mathbf{x}_k)\, p(\mathbf{x}_k|\,\mathcal{Z}^{k-1})$$

$$= \int d\mathbf{x}_k\, \underbrace{\mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k)}_{\text{likelihood: sensor model}}\, \underbrace{\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})}_{\text{prediction at time } t_k}$$

$$= \mathcal{N}(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_{k|k-1},\, \mathbf{S}_{k|k-1}) \qquad \text{(product formula)}$$

innovation: $\boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$,

innovation covariance: $\mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_k$

expectation gate: $\boldsymbol{\nu}_{k|k-1}^\top \mathbf{S}_{k|k-1}^{-1} \boldsymbol{\nu}_{k|k-1} \leq \lambda(P_c)$

MAHALANOBIS ellipsoid containing $\mathbf{z}_k$ with certain probability P_c

Choose $\lambda(P_c)$ (“gating parameter”) properly!

Can be looked up in a χ^2-table - discussed later!

Sensor data: range, azimuth, range-rate

Coordinates: Sensor data → *polar*, object evolution → *Cartesian*

Dynamics system: $\mathbf{x}^d = (x, y, \dot{x}, \dot{y})$

Sensor system: $\mathbf{x}^s = (r, \varphi, \dot{r}, \dot{\varphi})$

$$
\begin{array}{ccccc}
p\left(\mathbf{x}^d_{k-1} \middle| \mathcal{Z}^{k-1}\right) & \xrightarrow[\text{Dynamics}]{p\left(\mathbf{x}^d_k \middle| \mathbf{x}^d_{k-1}\right)} & p\left(\mathbf{x}^d_k \middle| \mathcal{Z}^{k-1}\right) & & p\left(\mathbf{x}^d_k \middle| \mathcal{Z}^{k}\right) \\
\uparrow \mathbf{t}_{d \leftarrow s} & & \downarrow \mathbf{t}_{s \leftarrow d} & & \uparrow \mathbf{t}_{d \leftarrow s} \\
p\left(\mathbf{x}^s_{k-1} \middle| \mathcal{Z}^{k-1}\right) & & p\left(\mathbf{x}^s_k \middle| \mathcal{Z}^{k-1}\right) & \xrightarrow[\text{Sensor}]{p\left(\mathbf{z}_k \middle| \mathbf{x}^s_k\right)} & p\left(\mathbf{x}^s_k \middle| \mathcal{Z}^{k}\right) \\
\text{scan } k-1 & & \text{scan } k & & \text{scan } k
\end{array}
$$

non-linear **coordinate transformations:**

$$
\mathbf{t}_{d \leftarrow s}[\mathbf{x}^s] = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r \cos\varphi \\ r \sin\varphi \\ \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \end{pmatrix} \qquad
\mathbf{t}_{s \leftarrow d}[\mathbf{x}^d] = \begin{pmatrix} r \\ \varphi \\ \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \arctan y/x \\ (x\dot{x}+y\dot{y})/\sqrt{x^2+y^2} \\ (x\dot{y}-y\dot{x})/(x^2+y^2) \end{pmatrix}
$$

Extended *Kalman* filter: be wise - linearize!

non-lineare transformations: Taylor expansion up to 1st order

around $\mathbf{x}^s_{k|k}$ (filtering): $\mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_k] \approx \mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}]\,(\mathbf{x}^s_k - \mathbf{x}^s_{k|k})$

mit: $\mathbf{T}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] = \partial \mathbf{t}_{d\leftarrow s}[\mathbf{x}^s_{k|k}] / \partial \mathbf{x}^s_{k|k}$ (Jacobian)

around $\mathbf{x}^d_{k|k-1}$ (Prediction): $\mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_k] \approx \mathbf{t}_{s\leftarrow d}[\mathbf{x}^d_{k|k-1}] + \mathbf{T}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}]\,(\mathbf{x}^d_k - \mathbf{x}^d_{k|k-1})$

with: $\mathbf{T}_{s\leftarrow d} = \partial \mathbf{t}_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}] / \partial \mathbf{x}^d_{k|k-1}$

affine transformation of Gaussian random variables:

$$\mathcal{N}\big(x;\, \mathbf{x},\, \mathbf{X}\big) \xrightarrow{y=\mathbf{a}+\mathbf{A}x} \mathcal{N}\big(y;\, \mathbf{a}+\mathbf{A}\mathbf{x},\, \mathbf{A}\mathbf{X}\mathbf{A}^\top\big)$$

Exercise 5.3 (voluntary) Calculate Jacobians $\mathbf{T}_{d\leftarrow s}$ and $\mathbf{T}_{s\leftarrow d}$.

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)$$

Retrodiction: How to calculate the pdf $p(\mathrm{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathrm{x}_l|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l, \mathrm{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big)\ \underbrace{p\big(\mathrm{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

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Retrodiction: How to calculate the pdf $p(\mathrm{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathrm{x}_l|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l, \mathrm{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big)\; \underbrace{p\big(\mathrm{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

$$p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big) = \frac{p(Z_k, \dots, Z_{l+1}|\mathrm{x}_{l+1}, \mathrm{x}_l, \mathcal{Z}^l)\; p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)}{\int d\mathrm{x}_l\; p(Z_k, \dots, Z_{l+1}|\mathrm{x}_{l+1}, \mathrm{x}_l, \mathcal{Z}^l)\; p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)} = p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big)$$

Retrodiction: How to calculate the pdf $p(\mathrm{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathrm{x}_l|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l, \mathrm{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathrm{x}_{l+1}\, p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big)\ \underbrace{p\big(\mathrm{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

$$p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^k\big) = p\big(\mathrm{x}_l|\,\mathrm{x}_{l+1}, \mathcal{Z}^l\big) = \frac{p\big(\mathrm{x}_{l+1}|\,\mathrm{x}_l\big)\ p\big(\mathrm{x}_l|\,\mathcal{Z}^l\big)}{\int d\mathrm{x}_l\ \underbrace{p\big(\mathrm{x}_{l+1}|\,\mathrm{x}_l\big)}_{\text{dynamics model}}\ \underbrace{p\big(\mathrm{x}_l|\,\mathcal{Z}^l\big)}_{\text{filtering } t_l}}$$

Retrodiction: How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p\big(\mathbf{x}_l|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, p\big(\mathbf{x}_l, \mathbf{x}_{l+1}|\,\mathcal{Z}^k\big) = \int d\mathbf{x}_{l+1}\, \frac{p\big(\mathbf{x}_{l+1}|\,\mathbf{x}_l\big)\, p\big(\mathbf{x}_l|\,\mathcal{Z}^l\big)}{\int d\mathbf{x}_l\, \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathbf{x}_l\big)}_{\text{dynamics model}}\, \underbrace{p\big(\mathbf{x}_l|\,\mathcal{Z}^l\big)}_{\text{filtering } t_l}}\, \underbrace{p\big(\mathbf{x}_{l+1}|\,\mathcal{Z}^k\big)}_{\text{retrodiction: } t_{l+1}}$$

- $p(\mathbf{x}_{l+1}|\mathcal{Z}^k)$ retrodiction: last iteration step
- $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ dynamic object behavior
- $p(\mathbf{x}_l|\mathcal{Z}^l)$ filtering at the time considered
- GAUSSians, GAUSSian mixtures: Exploit product formula!
- linear GAUSSian likelihood/dynamics: Rauch-Tung-Striebel smoothing

Fraunhofer FKIE

Exercise 5.4 Derive the Rauch-Tung-Striebel formulae

by using the Kalman filter assumptions

and the product formula (twice)!

retrodiction: $\mathcal{N}\big(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}\big) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}\big(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k}\big)$

$$\begin{aligned} \mathbf{x}_{l|k} &= \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^{\top}\mathbf{P}_{l+1|l}^{-1} \\ \mathbf{P}_{l|k} &= \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^{\top} \end{aligned}$$

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

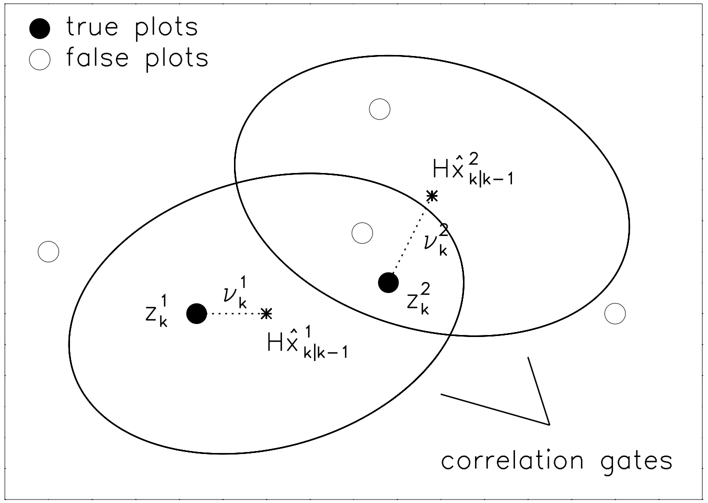
$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$

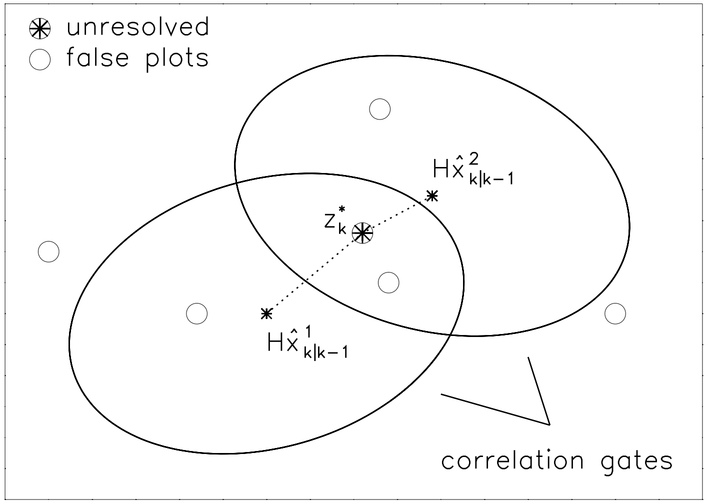
'KALMAN gain matrix'

retrodiction: $\mathcal{N}(\mathbf{x}_l;\, \mathbf{x}_{l|k},\, \mathbf{P}_{l|k}) \xleftarrow[\text{dynamics model}]{\text{filtering, prediction}} \mathcal{N}(\mathbf{x}_{l+1};\, \mathbf{x}_{l+1|k},\, \mathbf{P}_{l+1|k})$

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}), \qquad \mathbf{W}_{l|l+1} = \mathbf{P}_{l|l}\mathbf{F}_{l+1|l}^\top\mathbf{P}_{l+1|l}^{-1}$$
$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l})\mathbf{W}_{l|l+1}{}^\top$$

Sensor data of uncertain origin

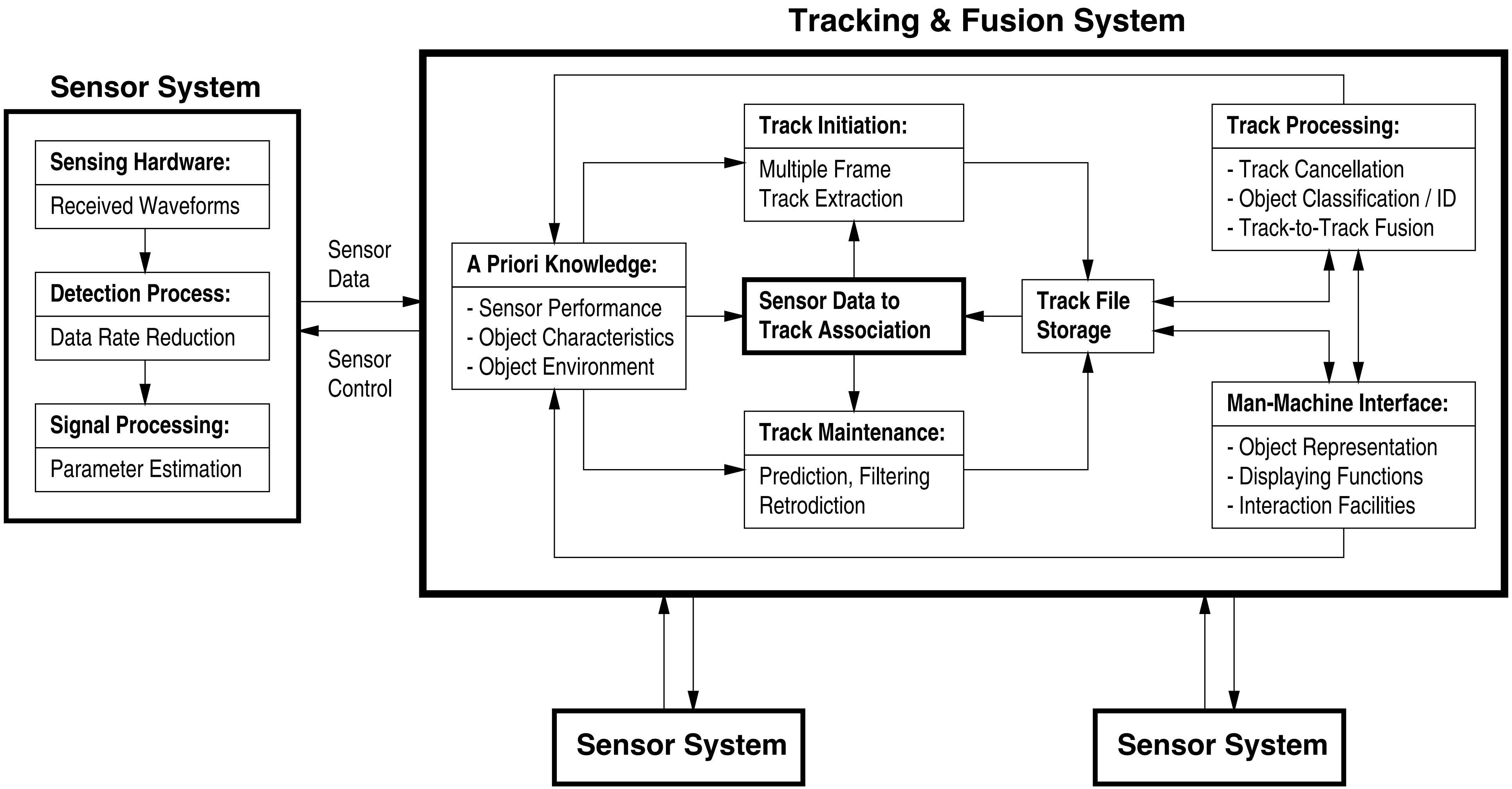




- prediction: $\mathbf{x}_{k|k-1}$, $\mathbf{P}_{k|k-1}$ (dynamics)
- innovation: $\boldsymbol{\nu}_k = \mathbf{z}_k - \mathbf{H}\mathbf{x}_{k|k-1}$, white
- Mahalanobis norm: $||\boldsymbol{\nu}_k||^2 = \boldsymbol{\nu}_k^\top \mathbf{S}_k^{-1} \boldsymbol{\nu}_k$
- expected plot: $\mathbf{z}_k \sim N(\mathbf{H}\mathbf{x}_{k|k-1}, \mathbf{S}_k)$
- $\boldsymbol{\nu}_k \sim N(0, \mathbf{S}_k)$, $\mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^\top + \mathbf{R}$
- gating: $||\boldsymbol{\nu}_k|| < \lambda$, $P_c(\lambda)$ correlation prob.

missing/false plots, measurement errors, scan rate, agile targets: large gates

A Generic Tracking and Sensor Data Fusion System



Fraunhofer FKIE

Description of the Detection Process

Detector: receives signals and decides on object existence

Processor: processes detected signals and produces measurements

: detector detects an object

D: object actually existent

Fraunhofer FKIE

Description of the Detection Process

Detector: receives signals and decides on object existence

Processor: processes detected signals and produces measurements

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D: object actually existent

error of 1. kind: $P_{\mathrm{I}} = P(\neg|D)$

error of 2. kind: $P_{\mathrm{II}} = P(|\neg D)$

measure of detection performance: $P_D = P(|D)$

detector properties characterized by two parameters:

– detection probability $P_D = 1 - P_{\mathrm{I}}$

– false alarm probability $P_F = P_{\mathrm{II}}$

Description of the Detection Process

Detector: receives signals and decides on object existence

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measure of detection performance: $P_D = P(|D)$

detector properties characterized by two parameters:

- detection probability $P_D = 1 - P_{\text{I}}$
- false alarm probability $P_F = P_{\text{II}}$

example (Swerling I model):	$P_D = P_D(P_F, \text{SNR}) = P_F^{1/(1+\text{SNR})}$

detector design: Maximize detection probability P_D
for a given, predefined false alarm probability P_F!

Fraunhofer FKIE

ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

$n_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$!

- E_0: the object was not detected; n_k false data in the Field of View (FoV)
- E_j, $j = 1, \ldots, n_k$: Object detected; $\mathbf{z}_k^j$ is object measurement; $n_k - 1$ false measurements

Consider the interpretations in the likelihood function $p(Z_k, n_k|\mathbf{x}_k)$!

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$n_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$!

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Consider the interpretations in the likelihood function $p(Z_k, n_k|\mathbf{x}_k)$!

$$p(Z_k, n_k|\mathbf{x}_k) = p(Z_k, n_k, \neg D|\mathbf{x}_k) + p(Z_k, n_k, D|\mathbf{x}_k) \qquad D = \text{“object was detected”}$$

ambiguous sensor data ($P_D < 1$, $\rho_F > 0$)

$n_k + 1$ possible interpretations of the sensor data $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{n_k}$!

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Consider the interpretations in the likelihood function $p(Z_k, n_k|\mathbf{x}_k)$!

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sensor parameter: detection probability P_D

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false measurements: Poisson distributed in #, uniformly distributed in the FoV

Fraunhofer FKIE

Modeling of False Measurements (FM)

- **Probability of having n FM:** $p_F(n)$
  - mean number of FM in the 'Field of View' (FoV) of a sensor:

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normalization: $\sum_{n=0}^{\infty} p_F(n) = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = \mathrm{e}^{-\bar{n}}\, \mathrm{e}^{\bar{n}} = 1$

expectation: $\mathbb{E}[n] = \mathrm{e}^{-\bar{n}} \sum_{n=0}^{\infty} n \frac{\bar{n}^n}{n!} = \mathrm{e}^{-\bar{n}} \sum_{n=1}^{\infty} n \frac{\bar{n}^n}{n!} = \bar{n}\mathrm{e}^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^{n-1}}{(n-1)!} = \bar{n}$

variance: $\mathbb{V}[n] = \mathbb{E}[(n-\bar{n})^2] = \mathbb{E}[n^2] - \bar{n}^2 = \ldots \text{exercise!} \ldots = \bar{n}$

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  - uniformly distributed in the FoV: $p(\mathbf{z}_i^f|\mathsf{FoV}) = |\mathsf{FoV}|^{-1}$ (often!)

Fraunhofer FKIE

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$$= |\text{FoV}|^{-n_k}\, p_F(n_k)\, (1 - P_D) + P_D \sum_{j=1}^{n_k} \underbrace{p(Z_k|n_k, j, D, \mathbf{x}_k)}_{|\text{FoV}|^{-(n_k-1)} N(\mathbf{z}_k^j; \mathbf{H}\mathbf{x}_k, \mathbf{R})}\; \underbrace{p(j|n_k, D)}_{=1/n_k}\; \underbrace{p(n_k|D)}_{=p_F(n_k-1)}$$

Insert Poisson distribution: $p_F(n_k) = \frac{(\rho_F|\text{FoV}|)^{-n_k}}{n_k!}\, \mathrm{e}^{-\rho_F|\text{FoV}|}$

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$$= \frac{e^{-\rho_F|\text{FoV}|}}{n_k!}\, \rho_F^{n_k-1}\Big((1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big)\Big)$$

Fraunhofer FKIE

Likelihood Functions

The likelihood function answers the question:

What does the sensor tell about the state $\mathbf{x}$ of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1$, $\rho_F = 0$

  at each time one measurement: $$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k; \mathbf{H}\mathbf{x}_k, \mathbf{R})$$

- **real conditions, one object:** $P_D < 1$, $\rho_F > 0$

  at each time n_k measurements $Z_k = \{\mathbf{z}_k^1, \ldots, \mathbf{z}_k^{n_k}\}$!

$$p(Z_k, n_k|\mathbf{x}_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}\big(\mathbf{z}_k^j;\ \mathbf{H}\mathbf{x}_k,\ \mathbf{R}\big)$$

Bayes Filtering for: $P_D < 1, \rho_F > 0$, well-separated objects

state $\mathbf{x}_k$, ***current* data** $Z_k = \{\mathbf{z}_k^j\}_{j=1}^{m_k}$, ***accumulated* data** $\mathcal{Z}^k = \{Z_k, \mathcal{Z}^{k-1}\}$

interpretation hypotheses E_k **for** Z_k

$$\left.\begin{array}{l}\text{object not detected, } 1 - P_D \\ \mathbf{z}_k \in Z_k \text{ from object, } P_D\end{array}\right\} m_k + 1 \text{ interpretations}$$

interpretation histories H_k **for** $\mathcal{Z}^k$

- tree structure: $H_k = (E_{H_k}, H_{k-1}) \in \mathcal{H}^k$
- current: E_{H_k}, *pre*histories: H_{k-i}

$$p(\mathbf{x}_k | \mathcal{Z}^k) = \sum_{H_k} p(\mathbf{x}_k, H_k | \mathcal{Z}^k) = \sum_{H_k} \underbrace{p(H_k | \mathcal{Z}^k)}_{\text{weight!}} \underbrace{p(\mathbf{x}_k | H_k, \mathcal{Z}^k)}_{\textit{given } H_k\text{: unique}}$$

'mixture' density

Closer look: $P_D < 1, \rho_F > 0$, well-separated targets

filtering (at time t_{k-1}): $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \sum_{H_{k-1}} p_{H_{k-1}} \, \mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{H_{k-1}},\, \mathbf{P}_{H_{k-1}}\big)$

prediction (for time t_k):

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \qquad \text{(MARKOV model)}$$

$$= \sum_{H_{k-1}} p_{H_{k-1}}\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}\mathbf{x}_{H_{k-1}},\, \mathbf{F}\mathbf{P}_{H_{k-1}}\mathbf{F}^\top + \mathbf{D}\big) \qquad \text{(IMM also possible)}$$

measurement likelihood:

$$p(Z_k, m_k|\mathbf{x}_k) = \sum_{j=0}^{m_k} p(\mathcal{Z}_k|E_k^j, \mathbf{x}_k, m_k)\, P(E_k^j|x_k, m_k) \qquad (E_k^j\text{: interpretations})$$

$$\propto (1 - P_D)\,\rho_F + P_D \sum_{j=1}^{m_k} \mathcal{N}\big(\mathbf{z}_k^j;\, \mathbf{H}\mathbf{x}_k,\, \mathbf{R}\big) \quad (\mathbf{H},\, \mathbf{R},\, P_D,\, \rho_F)$$

filtering (at time t_k):

$$p(\mathbf{x}_k|\mathcal{Z}^k) \propto p(Z_k, m_k|\mathbf{x}_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \qquad \text{(BAYES' rule)}$$

$$= \sum_{H_k} p_{H_k}\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{H_k},\, \mathbf{P}_{H_k}\big) \qquad \text{(Exploit product formula)}$$

Fraunhofer FKIE