Kalman filter: linear Gaussian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\begin{aligned}
\mathbf{x}_{k|k-1} &= \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1} \\
\mathbf{P}_{k|k-1} &= \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}
\end{aligned}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\begin{aligned}
\mathbf{x}_{k|k} &= \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, & \boldsymbol{\nu}_{k|k-1} &= \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1} \\
\mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, & \mathbf{S}_{k|k-1} &= \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k \\
\mathbf{W}_{k|k-1} &= \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} & &\text{'Kalman gain matrix'}
\end{aligned}$$

Exercise 3.1

Consider an object that moves in two dimensions on the following trajectory:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} \sin(\omega t) \\ \sin(2\omega t) \end{pmatrix} \quad \text{with} \quad A = \frac{v^2}{q}, \quad \omega = \frac{q}{2v}$$

and speed ans acceleration parameters: $v = 300\,\frac{\text{m}}{\text{s}}$, $q = 9\,\frac{\text{m}}{\text{s}^2}$!

1. Plot the trajectory. Why is it periodical? What is its period $T = T(v, q)$?
2. Show for the velocity and acceleration vector:

$$\dot{\mathbf{r}}(t) = v \begin{pmatrix} \cos(\omega t)/2 \\ \cos(2\omega t) \end{pmatrix}, \quad \ddot{\mathbf{r}}(t) = -q \begin{pmatrix} \sin(\omega t)/4 \\ \sin(2\omega t) \end{pmatrix}!$$

3. Calculate for each instance of time t the tangential and normal vectors in $\mathbf{r}(t)$:

$$\mathbf{t}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}, \quad \mathbf{n}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \begin{pmatrix} -\dot{y}(t) \\ \dot{x}(t) \end{pmatrix}!$$

4. Plot $|\dot{\mathbf{r}}(t)|$, $|\ddot{\mathbf{r}}(t)|$, $\ddot{\mathbf{r}}(t)\mathbf{t}(t)$ and $\ddot{\mathbf{r}}(t)\mathbf{n}(t)$ üover a period T!
5. Discuss the temporal behaviour based on the trajectory $\mathbf{r}(t)$! What are the maximum speeds and accelerations, v_{max}, q_{max}?

Exercise 3.2

Consider a sensor in the coordinate origin with constant revisit interval Δt = 5 s measuring the Cartesian position $\mathbf{r}_k = \mathbf{r}(t_k)$ of the objects at times $t_k = k\Delta t$, $k = 0, 1, \ldots$ with Gaussian measurement errors (independent, identically distributed: i.i.d.), covariance matrix $\mathbf{R} = \sigma^2 \mathbf{I}$, σ = 50 m. **Mandatory!**

1. Simulate with a random generator measurements $\mathbf{z}_k = (z_k^x, z_k^y)$ according to:

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \sigma \begin{pmatrix} u_k^x \\ u_k^y \end{pmatrix}, \qquad \text{mit} \quad \mathbf{x}_k = \left(\mathbf{r}(t_k)^\top, \dot{\mathbf{r}}(t_k)^\top, \ddot{\mathbf{r}}(t_k)^\top\right)^\top,$$

$$\mathbf{H} = (\mathbf{I}, \mathbf{O}, \mathbf{O}), \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_k^x, u_k^y \sim N(0, 1)$$

   and plot the time series over the trajectory $\mathbf{r}(t)$!

2. Initiate the Kalman recursion by the a priori knowledge:

$$p(\mathbf{x}_0) = \mathcal{N}\left(\mathbf{x}_0;\, \mathbf{x}_{0|0}, \mathbf{P}_{0|0}\right), \quad \mathbf{x}_{0|0} = \begin{pmatrix} \mathbf{z}_0 \\ \mathbf{o} \\ \mathbf{o} \end{pmatrix}, \quad \mathbf{P}_{0|0} = \begin{pmatrix} \mathbf{R} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & v_{\text{max}}^2 \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & q_{\text{max}}^2 \mathbf{I} \end{pmatrix}, \quad \mathbf{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3. Realize the Kalman recursion as a program and calculate the prediction $\mathbf{x}_{k|k-1}$, $\mathbf{P}_{k|k-1}$ and filtering $\mathbf{x}_{k|k}$, $\mathbf{P}_{k|k}$ für $k = 1, 2, \ldots$. Use the dynamics model:

$$\mathbf{F} = \begin{pmatrix} \mathbf{I} & \Delta t\, \mathbf{I} & \frac{1}{2}\Delta t^2\, \mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta t\, \mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-\Delta t/\theta}\, \mathbf{I} \end{pmatrix}, \quad \mathbf{D} = \Sigma^2 (1 - \mathrm{e}^{-2\Delta t/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \Sigma = q_{\text{max}}, \ \theta = 60\text{s}$$

4. Plot measurements, predictions, filtering with corresponding error ellipses (eigen values/vectors!) and compare with the truth! Play with the parameters!

Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

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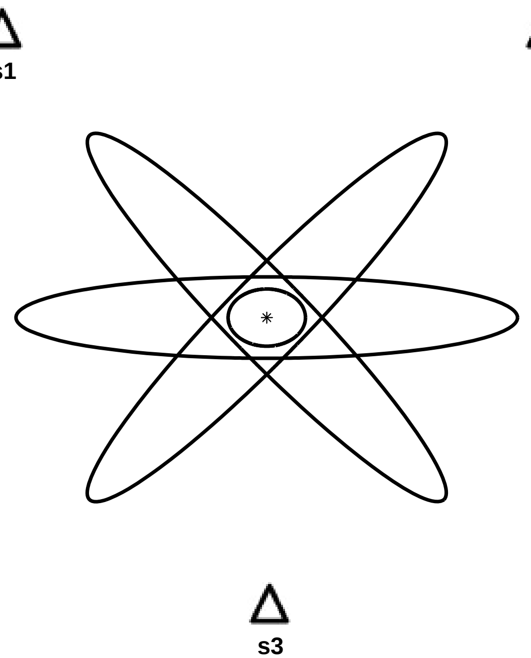
a closer look: The error of each measurement $\mathbf{z}_i$ is described by a related measurement error *covariance matrix* $\mathbf{R}_i$ ('error ellipsoids'). In 2 dimensions:



$\mathbf{R}_i$ can strongly depend on the underlying senor-to-target geometry!

Fraunhofer FKIE

Simplified: Range, Azimuth Measurements

- **measurements in polar coordinates:**

  $\mathbf{z}_k = (r_k, \varphi_k)^\top$, measurement error: $\mathbf{R} = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\varphi^2 \end{pmatrix}$, r, φ independent

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Likelihood function in polar coordinates:

$$p(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k;\, \mathbf{x}_k^p,\, \mathbf{R}^p)$$

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- **What is the likelihood function in Cartesian coordinates?**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix}$$

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- **in Cartesian coord.: expand around $\mathbf{r}_{k|k-1} = (r_{k|k-1}, \varphi_{k|k-1})^\top$:**

$$\mathbf{t}[\mathbf{z}_k] = r_k \begin{pmatrix} \cos\varphi_k \\ \sin\varphi_k \end{pmatrix} \approx \mathbf{t}[\mathbf{r}_{k|k-1}] + \mathbf{T}\,(\mathbf{z}_k - \mathbf{r}_{k|k-1})$$

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- **affine transform of GAUSSian random variables:**

$$\mathcal{N}\big(\mathbf{z};\, \mathbf{x},\, \mathbf{R}\big) \xrightarrow{\mathbf{z}'=\mathbf{t}+\mathbf{T}\mathbf{z}} \mathcal{N}\big(\mathbf{z}';\, \mathbf{t}+\mathbf{T}\mathbf{x},\, \mathbf{T}\mathbf{R}\mathbf{T}^\top\big)$$

Simplified: Range, Azimuth Measurements

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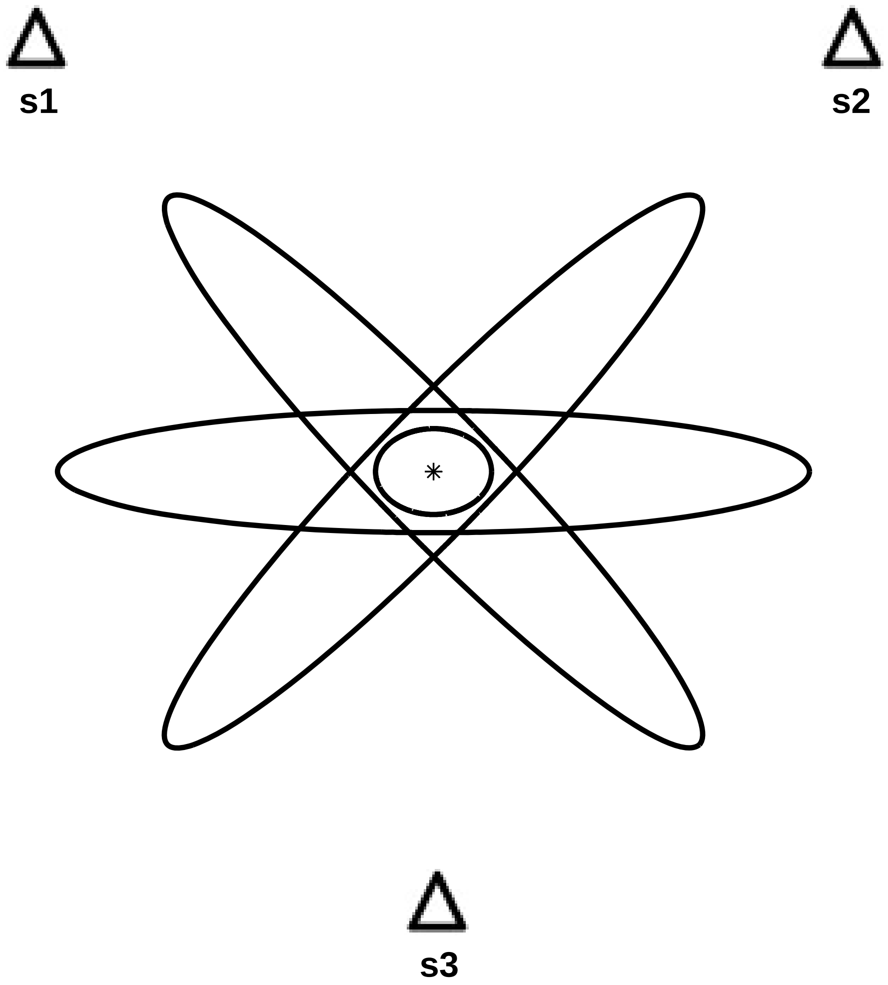
- **Cartesian error covariance (time dependent):**

$$\mathbf{T}\mathbf{R}\mathbf{T}^\top = \mathbf{D}_\varphi \mathbf{S}_r\, \mathbf{R}\, \mathbf{S}_r D_\varphi^\top = \mathbf{D}_\varphi \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & (r\sigma_\varphi)^2 \end{pmatrix} \mathbf{D}_\varphi^\top$$

- **sensor fusion: sensor-to-target-geometry enters into $\mathbf{T}\mathbf{R}\mathbf{T}^\top$**



sensor fusion: sensor-to-target-geometry enters into $\mathrm{TRT}^\top$

Modelling of the Measurement Process

- **What tells the measurement about the object?**

  functional relation between measurement $\mathbf{z}_k = (z_k^1, \ldots, z_k^m)$ at time t_k and the object state $\mathbf{x}_k = (x_k^1, \ldots, x_k^n)^\top$ at the same time

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  Example for the 2D object state $\mathbf{x}_k = (x_k^1, x_k^2)^\top$:

  – x_1 component measured: $\mathbf{z}_k = \mathbf{H}\mathbf{x}_k, \quad \mathbf{H} = (\,1\;0\,)$ linear function

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  – range/distance: $\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k), \quad \mathbf{h}(\mathbf{x}_k) = \begin{pmatrix} \sqrt{(x_k^1)^2 + (x_k^2)^2} \\ \tan^{-1} x_k^2/x_k^1 \end{pmatrix}$ non-linear!

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- **What is the quality of the measurement?**

  reasonable: measurement error $\mathbf{u}_k = \mathbf{z}_k - \mathbf{H}\mathbf{x}_k$ normally distributed with:

$$\mathbb{E}[\mathbf{u}_k] = 0,\ \mathbb{C}[\mathbf{u}_k] = \begin{pmatrix} \mathbb{E}[(u_k^1)^2] & \mathbb{E}[u_k^1 u_k^2] \\ \mathbb{E}[u_k^2 u_k^1] & \mathbb{E}[(u_k^2)^2] \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \mathbb{E}[u_k^1 u_k^2] \\ \mathbb{E}[u_k^2 u_k^1] & \sigma_2^2 \end{pmatrix} = \mathbf{R}$$

  uncorrelated: $\mathbb{E}[u_k^1 u_k^2] = 0$; Gaussian: uncorrelated $\Rightarrow$ independent (not i.g.!)
  Standard deviations σ appear on the diagonal of the covariance matrix.

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Idealisized measurement process

- **linear measurement equation:**

  $$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{u}_k, \quad p(\mathbf{u}_k) = \mathcal{N}\Big(\mathbf{u}_k;\, \mathbf{o},\, \mathbf{R}_k\Big)$$

  – to be measured: *linear* functions of the object state
  – measurement error: biasfree, Gaussian distrib. independent for different t_k
  – $\mathbf{y}_k = \mathbf{z}_k - \mathbf{H}_k \mathbf{x}_k$ has the pdf: $p(\mathbf{y}_k) = p(\mathbf{u}_k)$

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- **Approach for the requested pdf ('likelihood fkt.):**

  $$\boxed{p\big(\mathbf{z}_k \big| \mathbf{x}_k\big) = \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k \mathbf{x}_k,\, \mathbf{R}_k\big)}$$

- **Example: position measurement**

  $$\mathbf{H}_k = (\mathbf{I}, \mathbf{O}, \mathbf{O}), \quad \mathbf{H}_k \mathbf{x}_k = \mathbf{r}_k$$

  $\mathbf{R}_k = \text{diag}[\sigma_x^2, \sigma_y^2, \sigma_z^2],\ \sigma_x:$ measurement error

Object evolution: Gauß-Markov process

- **linear evolution equation:** $\mathbf{x}_k = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1} + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{o}, \mathbf{D}_{k|k-1})$

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- **Very simple example:**

  Object on a strait line: 2D state $\mathbf{x}_k = (x_k, \dot{x}_k)^\top$

  simple approach:
  $$\begin{aligned} x_k &= x_{k-1} + \Delta t\, \dot{x}_{k-1} & \Delta t = t_k - t_{k-1} \\ \dot{x}_k &= \dot{x}_{k-1} + v & v \sim N(0, D) \end{aligned}$$

  we thus have: $\mathbf{x}_k = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \mathbf{x}_{k-1} + \mathbf{v}, \quad \mathbf{v} \sim N(0, \mathbf{D}), \ \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$

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- **Requested: Markov property!**

$$p\big(\mathbf{x}_k \big| \mathbf{x}_{k-1}, \mathbf{x}_{k-2}, \ldots, \mathbf{x}_1\big) \stackrel{!}{=} p\big(\mathbf{x}_k \big| \mathbf{x}_{k-1}\big) \stackrel{!}{=} \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}_{k|k-1}\big)$$

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therefore:
$$\begin{aligned} p(\mathbf{x}_k) &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k, \mathbf{x}_{k-1}, \dots \mathbf{x}_1) \\ &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}, \dots \mathbf{x}_1) \\ &= \int d\mathbf{x}_{k-1} \cdots \int d\mathbf{x}_1\, p(\mathbf{x}_k|\mathbf{x}_{k-1}) \dots p(\mathbf{x}_2|\mathbf{x}_1)\, p(\mathbf{x}_1) \end{aligned}$$

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- **a more realistic model:**

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} & \frac{1}{2}(t_k - t_{k-1})^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-(t_k - t_{k-1})/\theta}\,\mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \mathrm{diag}[1, 1, 1]$$

$$\mathbf{D}_{k|k-1} = \Sigma^2 (1 - \mathrm{e}^{-2(t_k - t_{k-1})/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{O} = \mathrm{diag}[0, 0, 0]$$

There are many different evolution models adapted to particular problems!

Fraunhofer FKIE

- **a more realistic model:**

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} & \frac{1}{2}(t_k - t_{k-1})^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & (t_k - t_{k-1})\,\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-(t_k - t_{k-1})/\theta}\,\mathbf{I} \end{pmatrix}, \quad \mathbf{I} = \mathrm{diag}[1, 1, 1]$$

$$\mathbf{D}_{k|k-1} = \Sigma^2 (1 - \mathrm{e}^{-2(t_k - t_{k-1})/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{O} = \mathrm{diag}[0, 0, 0]$$

There are many different evolution models adapted to particular problems!

Show for the acceleration process:

$$\mathbb{E}[\ddot{\mathbf{r}}_k] = 0, \quad \mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top] = \Sigma^2\, \mathrm{e}^{-(t_k - t_l)/\theta}\, \mathbf{I}, \; l \leq k$$

Exercise 4.1 (voluntary)

θ: maneuver correlation time, Σ: limiting acceleration

$\mathbb{E}[\ddot{\mathbf{r}}_k \ddot{\mathbf{r}}_l^\top]$ is called ‘auto correlation function’.

Fraunhofer FKIE

Another popular model for object evolutions

Piecewise Constant White Acceleration Model

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

For known $\mathbf{x}_{k-1}$ and without external influences we have with ΔT_k:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{r}_{k-1} + \Delta T_k \, \dot{\mathbf{r}}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \Delta T_k \, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1} \mathbf{x}_{k-1}$$

Another popular model for object evolutions

Piecewise Constant White Acceleration Model

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

For known $\mathbf{x}_{k-1}$ and without external influences we have with $\Delta T_k = t_k - t_{k-1}$:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{I} & \Delta T_k \, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1} \mathbf{x}_{k-1}$$

Assume during the interval ΔT_k a constant acceleration $\mathbf{a}_k$ causing the state evolution:

$$\begin{pmatrix} \frac{1}{2} \Delta T_k^2 \, \mathbf{I} \\ \Delta T_k \, \mathbf{I} \end{pmatrix} \mathbf{a}_k =: \mathbf{G}_k \mathbf{a}_k.$$

Another popular model for object evolutions

Piecewise Constant White Acceleration Model

Consider state vectors: $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top)^\top$ (position, velocity)

For known $\mathbf{x}_{k-1}$ and without external influences we have with $\Delta T_k = t_k - t_{k-1}$:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{I} & \Delta T_k \, \mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{r}_{k-1} \\ \dot{\mathbf{r}}_{k-1} \end{pmatrix} =: \mathbf{F}_{k|k-1} \mathbf{x}_{k-1}$$

Assume during the interval ΔT_k a constant acceleration $\mathbf{a}_k$ causing the state evolution:

$$\begin{pmatrix} \frac{1}{2} \Delta T_k^2 \, \mathbf{I} \\ \Delta T_k \, \mathbf{I} \end{pmatrix} \mathbf{a}_k =: \mathbf{G}_k \mathbf{a}_k.$$

Let $\mathbf{a}_k$ be a Gaussian RV with pdf: $p(\mathbf{a}_k) = \mathcal{N}\big(\mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2 \mathbf{I}\big)$, we therefore have:

$$p(\mathbf{G}_k \mathbf{a}_k) = \mathcal{N}\big(\mathbf{G}_k \mathbf{a}_k;\, \mathbf{o},\, \Sigma_k^2 \mathbf{G}_k \mathbf{G}_k^\top\big).$$

Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\Big(\mathbf{x}_k;\ \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\ \mathbf{D}_{k|k-1}\Big)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T_k^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

Therefore: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)$ with

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T_k^3\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\,\mathbf{I} \end{pmatrix}$$

Exercise 4.2 Consider $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$ (position, velocity, acceleration)

Show that $\mathbf{F}_{k|k-1}$ and $\mathbf{D}_{k|k-1} = \Sigma_k^2 \mathbf{G}_k \mathbf{G}_k^\top$ (constant acceleration rates) are given by:

$$\mathbf{F}_{k|k-1} = \begin{pmatrix} \mathbf{I} & \Delta T_k\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta T_k\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{I} \end{pmatrix}, \quad \mathbf{D}_{k|k-1} = \Sigma_k^2 \begin{pmatrix} \frac{1}{4}\Delta T_k^4\,\mathbf{I} & \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \frac{1}{2}\Delta T_k^2\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^3\,\mathbf{I} & \Delta T_k^2\mathbf{I} & \Delta T_k\,\mathbf{I} \\ \frac{1}{2}\Delta T_k^2\,\mathbf{I} & \Delta T_k\mathbf{I} & \mathbf{I} \end{pmatrix}$$

with $\Delta T_k = t_k - t_{k-1}$. Reasonable choice: $\frac{1}{2}q_{\text{max}} \leq \Sigma_k \leq q_{\text{max}}$

Kalman filter: general properties

$$
\begin{aligned}
\mathbf{x}_{k|k} &= \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\,\boldsymbol{\nu}_{k|k-1} \\
\mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}^{\top}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{S}_{k|k-1} &= \mathbf{H}_{k|k-1}\mathbf{P}_{k|k-1}\mathbf{H}_{k|k-1}^{\top} + \mathbf{R}_{k|k-1} \\
\mathbf{W}_{k|k-1} &= \mathbf{P}_{k|k-1}\mathbf{H}_{k|k-1}^{\top}\mathbf{S}_{k|k-1}
\end{aligned}
$$

- *all elements* of the density: estimate an quality measure
- *variable* update, *time*-dependent evolution, measurement error
- variable *type*: measurement matrix (e.g. incomplete measurements)
- low computational effort (e.g. analytic inversions)

Filtering step: alternative formulation

$$
\begin{aligned}
p\big(\mathbf{x}_k \big| \mathcal{Z}^k\big) &= p\big(\mathbf{x}_k \big| \mathbf{z}_k, \mathcal{Z}^{k-1}\big) && \text{(current measurement)} \\
&= \frac{p\big(\mathbf{z}_k \big| \mathbf{x}_k\big)\, p\big(\mathbf{x}_k \big| \mathcal{Z}^{k-1}\big)}{\int d\mathbf{x}_k\, p\big(\mathbf{z}_k \big| \mathbf{x}_k\big)\, p\big(\mathbf{x}_k \big| \mathcal{Z}^{k-1}\big)} && \text{(Bayes' rule)} \\
&= \frac{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}{\int d\mathbf{x}_k\, \underbrace{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)}_{\text{likelihood function}}\, \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}_{\text{prediction for } t_k}} \\
&= \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big) && \text{(product formula: 2. version!)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}_{k|k} &= \mathbf{P}_{k|k}^{-1}\big(\mathbf{P}_{k|k-1}^{-1}\mathbf{x}_{k|k-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{z}_k\big) \\
\mathbf{P}_{k|k}^{-1} &= \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^\top \mathbf{R}_k^{-1}\mathbf{H}
\end{aligned}
$$

inverse covariance matrices are called information matrices.

Fraunhofer FKIE