Recapitulation: Multivariate GAUSSian Pdf

– *wanted:* probabilities 'concentrated' around a center $\mathbf{x}$

– *quadratic distance:* $q(x) = \frac{1}{2}(x - \mathbf{x})\mathbf{P}^{-1}(x - \mathbf{x})^\top$

$q(x)$ defines an ellipsoid around $\mathbf{x}$, its volume and orientation being determined by a matrix $\mathbf{P}$ (symmetric: $\mathbf{P}^\top = \mathbf{P}$, positively definite: all eigenvalues > 0).

– *first attempt:* $p(x) = \mathrm{e}^{-q(x)} / \int dx\, \mathrm{e}^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x - \mathbf{x})^\top \mathbf{P}^{-1}(x - \mathbf{x})}$$

$$\mathbb{E}[x] = \mathbf{x},\ \ \mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P} \ \ \text{(covariance)}$$

– GAUSS*ian Mixtures:* $p(x) = \sum_i p_i\, \mathcal{N}(x;\, \mathbf{x}_i,\, \mathbf{P}_i)$ (weighted sums)

A (very!) useful product formula for GAUSSians

Exercise 3.1 (voluntary) Show:

$$\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\, \mathbf{y},\, \mathbf{P}\big) = \underbrace{\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S}\big)}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}\big(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top\big) \\ \mathcal{N}\big(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}\big) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Sketch of the proof:

- Interpret $\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\, \mathbf{y},\, \mathbf{P}\big)$ as a joint pdf $p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{z}, \mathbf{x})$.
- Show that $p(\mathbf{z}, \mathbf{x})$ is a GAUSSian: $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\Big(\big(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\big);\, \big(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\big),\, \big(\begin{smallmatrix}\mathbf{S} & \mathbf{H}\mathbf{P}\\ \mathbf{P}\mathbf{H}^\top & \mathbf{P}\end{smallmatrix}\big)\Big)$.
- Calculate from $p(\mathbf{z}, \mathbf{x})$ the marginal and conditional pdfs $p(\mathbf{z})$ and $p(\mathbf{x}|\mathbf{z})$.
- From $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) = p(\mathbf{x}, \mathbf{z})$ we obtain the result.

Other (very!) useful formulae needed for exercise 3.1:

The inverse of a partitioned symmetric matrix is:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{C}^\top\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}$$

with: $\mathbf{S} = \mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C}$

"*Schur complement* of the matrix $\mathbf{A}$"

proof: check it!

An alternative formulation:

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{T}^{-1} & -\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{C}^\top\mathbf{T}^{-1} & \mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{C}^\top\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix}$$

with: $\mathbf{T} = \mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top$

We in particular have the "matrix inversion lemma":

$$(\mathbf{A} - \mathbf{C}\mathbf{B}^{-1}\mathbf{C}^\top)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}(\mathbf{B} - \mathbf{C}^\top\mathbf{A}^{-1}\mathbf{C})^{-1}\mathbf{C}^\top\mathbf{A}^{-1}$$

- $p(\mathbf{x}_k|\mathcal{Z}^{k-1})$ is a *prediction* of the target state at time t_k based on all measurements in the *past*.

$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k, \mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \quad \text{marginal pdf}$$

$$= \int d\mathbf{x}_{k-1}\, \underbrace{p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1})}_{\text{object dynamics!}}\, \underbrace{p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}_{\text{idea: iteration!}} \quad \text{notion of a conditional pdf}$$

often: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1})$ (MARKOV)

sometimes: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}}_{\text{deterministic}}\mathbf{x}_{k-1},\, \underbrace{\mathbf{D}_{k|k-1}}_{\text{random}}\big)$ (linear GAUSS-MARKOV)

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- $p(Z_k, m_k|\mathbf{x}_k) \propto \ell(\mathbf{x}_k; Z_k, m_k)$ describes, what the *current* sensor output Z_k, m_k can say about the current target state $\mathbf{x}_k$ and is called *likelihood function*.

sometimes: $\ell(\mathbf{z}_k; \mathbf{x}_k) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$ (1 target, 1 measurement)

iteration formula:

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}$$

GAUSSian transition pdf: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1})$

with: $\underbrace{\mathbf{F}_{k|k-1} \text{ (evolution matrix)}}_{\text{describes deterministic motion}}$, $\underbrace{\mathbf{D}_{k|k-1} \text{ (dynamics covariance matrix)}}_{\text{models random maneuvers}}$

GAUSSian posterior: $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1})$

GAUSSian transition pdf: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1})$

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$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \int d\mathbf{x}_{k-1}\; \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)}_{\text{dynamics model}}\; \underbrace{\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big)}_{\text{posterior at time } t_{k-1}}$$

GAUSSian transition pdf: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1})$

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GAUSSian posterior: $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1})$

$$
\begin{aligned}
p(\mathbf{x}_k|\mathcal{Z}^{k-1}) &= \int d\mathbf{x}_{k-1}\ \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)}_{\text{dynamics model}}\ \underbrace{\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big)}_{\text{posterior at time } t_{k-1}} \\
&= \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}}_{=:\mathbf{x}_{k|k-1}},\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}^{\top} + \mathbf{D}_{k|k-1}}_{=:\mathbf{P}_{k|k-1}}\big) \\
&\quad \times \underbrace{\int d\mathbf{x}_{k-1}\ \mathcal{N}\big(\mathbf{x}_{k-1};\, \ldots,\, \ldots\big)}_{=1 \quad \text{(normalization!)}} \qquad \text{(exploit product formula!)}
\end{aligned}
$$

GAUSSian transition pdf: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1})$

with: $\underbrace{\mathbf{F}_{k|k-1} \text{ (evolution matrix)}}_{\text{describes deterministic motion}}$, $\underbrace{\mathbf{D}_{k|k-1} \text{ (dynamics covariance matrix)}}_{\text{models random maneuvers}}$

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$$p(\mathbf{x}_k|\mathcal{Z}^{k-1}) = \int d\mathbf{x}_{k-1}\, \underbrace{\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1}\big)}_{\text{dynamics model}}\, \underbrace{\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big)}_{\text{posterior at time } t_{k-1}}$$

$$= \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}}_{=:\mathbf{x}_{k|k-1}},\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}^{\top} + \mathbf{D}_{k|k-1}}_{=:\mathbf{P}_{k|k-1}}\big)$$

$$\times \underbrace{\int d\mathbf{x}_{k-1}\, \mathcal{N}\big(\mathbf{x}_{k-1};\, \dots,\, \dots\big)}_{=1 \text{ (normalization!)}} \qquad \text{(exploit product formula!)}$$

$$= \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$$

Bayesian filtering update based on predictions (KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{\ell(\mathbf{x}_k; Z_k, m_k)\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)}$$

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$$= \frac{\ell(\mathbf{x}_k; Z_k, m_k)\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)}$$

Consider as a simple example $\ell(\mathbf{z}_k; Z_k, m_k) = \mathcal{N}\big(\mathbf{z}_k;\ \mathbf{H}_k\mathbf{x}_k,\ \mathbf{R}_k\big)$!

$$= \frac{\mathcal{N}\big(\mathbf{z}_k;\ \mathbf{H}_k\mathbf{x}_k,\ \mathbf{R}_k\big)\ \mathcal{N}(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1})}{\int d\mathbf{x}_k\ \mathcal{N}\big(\mathbf{z}_k;\ \mathbf{H}_k\mathbf{x}_k,\ \mathbf{R}_k\big)\ \mathcal{N}\big(\mathbf{x}_k;\ \mathbf{x}_{k|k-1},\ \mathbf{P}_{k|k-1}\big)}$$

Bayesian filtering update based on predictions (KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

$$= \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}$$

Consider as a simple example $\ell(\mathbf{x}_k; Z_k, m_k) = \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$!

$$= \frac{\mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}{\int d\mathbf{x}_k\, \mathcal{N}\big(\mathbf{z}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)\, \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)}$$

$$= \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$$ (due to the product formula)

with: $\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{x}_{k|k}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1})$, $\quad \mathbf{W}_{k|k} = \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}_{k|k}$ (KALMAN update)

$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k}\mathbf{W}_{k|k-1}$, $\quad \mathbf{S}_{k|k} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k$.

a first remark on initiation: $p(\mathbf{x}_0|\mathcal{Z}^0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, $\mathbf{P}_{0|0}$ 'large'

$$\mathbf{x}_{0|0} = \begin{pmatrix} \mathbf{r}_{0|0} \\ \dot{\mathbf{r}}_{0|0} \\ \ddot{\mathbf{r}}_{0|0} \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{P}_{0|0} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (v_{max})^2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (q_{max})^2\mathbf{1} \end{pmatrix}$$

position information: first measurement $\mathbf{z}_0$, ignorance = measurement error $\mathbf{R}$!

ignorance on velocity: sphere with radius v_{max} around zero
(= no information on direction, but on 'limits')

ignorance on acceleration: sphere with radius q_{max} around zero

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1} \qquad \text{'KALMAN gain matrix'}$$

A Useful Product Formula for Gaussians

$$\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\, \mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P}) = \underbrace{\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S})}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top) \\ \mathcal{N}(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Sketch of the proof:

- Interpret $\mathcal{N}(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R})\, \mathcal{N}(\mathbf{x};\, \mathbf{y},\, \mathbf{P})$ as a joint pdf $p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{z}, \mathbf{x})$.
- Show that $p(\mathbf{z}, \mathbf{x})$ is a Gaussian: $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\left(\left(\begin{smallmatrix}\mathbf{z}\\ \mathbf{x}\end{smallmatrix}\right);\, \left(\begin{smallmatrix}\mathbf{H}\mathbf{y}\\ \mathbf{y}\end{smallmatrix}\right),\, \left(\begin{smallmatrix}\mathbf{S} & \mathbf{H}\mathbf{P}\\ \mathbf{P}\mathbf{H}^\top & \mathbf{P}\end{smallmatrix}\right)\right)$.
- Calculate from $p(\mathbf{z}, \mathbf{x})$ the marginal and conditional pdfs $p(\mathbf{z})$ and $p(\mathbf{x}|\mathbf{z})$.
- From $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) = p(\mathbf{x}, \mathbf{z})$ we obtain the result.

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\big(\mathbf{x};\, \mathbb{E}[\mathbf{x}],\, \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\, \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\, \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\big(\mathbf{x};\, \mathbb{E}[\mathbf{x}],\, \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\, \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\, \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\, \mathbf{y},\, \mathbf{D}\big)$ mit $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{für } \mathbf{D} \to 0$$

Affine Transforms of GAUSSian Random Variables

$$\mathcal{N}\big(\mathbf{x};\ \mathbb{E}[\mathbf{x}],\ \mathbb{C}[\mathbf{x}]\big) \xrightarrow{\mathbf{y}=\mathbf{t}+\mathbf{T}\mathbf{x}} \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top\big)$$

$$p(\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{x},\mathbf{y}) = \int d\mathbf{x}\, p(\mathbf{y}|\mathbf{x})\, p(\mathbf{x}) = \int d\mathbf{x}\, \delta(\mathbf{y}-\mathbf{t}-\mathbf{T}\mathbf{x})\, p(\mathbf{x})$$

A possible representation: $\delta(\mathbf{x}-\mathbf{y}) = \mathcal{N}\big(\mathbf{x};\ \mathbf{y},\ \mathbf{D}\big)$ mit $\mathbf{D} \to \mathbf{O}$!

$$p(\mathbf{y}) = \int d\mathbf{x}\, \mathcal{N}(\mathbf{y};\mathbf{t}+\mathbf{T}\mathbf{x},\mathbf{D})\, \mathcal{N}(\mathbf{x};\mathbb{E}[\mathbf{x}],\mathbb{C}[\mathbf{x}]) \quad \text{for } \mathbf{D}\to 0$$

$$= \mathcal{N}\big(\mathbf{y};\ \mathbf{t}+\mathbf{T}\mathbb{E}[\mathbf{x}],\ \mathbf{T}\mathbb{C}[\mathbf{x}]\mathbf{T}^\top+\mathbf{D}\big) \qquad \text{for } \mathbf{D}\to 0; \quad \text{product formula!}$$

Also true if $\dim(\mathbf{x}) \neq \dim(\mathbf{y})$**!**

Bayesian Target Tracking: Basic Idea

Iterative updating of conditional probability densities!

kinematic target state $\mathbf{x}_k$ at time t_k, **accumulated sensor data** $\mathcal{Z}^k$

a priori knowledge: target evolution models, sensor model, further context information

- **prediction:** $p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1}) \xrightarrow[\text{context}]{\text{evolution model}} p(\mathbf{x}_k|\mathcal{Z}^{k-1})$
- **filtering:** $p(\mathbf{x}_k|\mathcal{Z}^{k-1}) \xrightarrow[\text{sensor model}]{\text{sensor data } Z_k} p(\mathbf{x}_k|\mathcal{Z}^k)$
- **retrodiction:** $p(\mathbf{x}_{l-1}|\mathcal{Z}^k) \xleftarrow[\text{evolution model}]{\text{filtering output}} p(\mathbf{x}_l|\mathcal{Z}^k)$

– ***finite mixture:*** **inherent ambiguity (sensor data, model, context)**
– ***optimal estimators:*** **e.g. minimum mean squared error (MMSE)**
– ***initiation of pdf iteration:*** **statistical decision on track existence**

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction: $\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$
$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering: $\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$
$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$
'KALMAN gain matrix'

Exercise 3.1

Consider an object that moves in two dimensions on the following trajectory:

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} \sin(\omega t) \\ \sin(2\omega t) \end{pmatrix} \quad \text{with} \quad A = \frac{v^2}{q}, \quad \omega = \frac{q}{2v}$$

and speed ans acceleration parameters: $v = 300\ \frac{\text{m}}{\text{s}}$, $q = 9\ \frac{\text{m}}{\text{s}^2}$!

1. Plot the trajectory. Why is it periodical? What is its period $T = T(v, q)$?
2. Show for the velocity and acceleration vector:

$$\dot{\mathbf{r}}(t) = v \begin{pmatrix} \cos(\omega t)/2 \\ \cos(2\omega t) \end{pmatrix}, \quad \ddot{\mathbf{r}}(t) = -q \begin{pmatrix} \sin(\omega t)/4 \\ \sin(2\omega t) \end{pmatrix}!$$

3. Calculate for each instance of time t the tangential and normal vectors in $\mathbf{r}(t)$:

$$\mathbf{t}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}, \quad \mathbf{n}(t) = \frac{1}{|\dot{\mathbf{r}}(t)|} \begin{pmatrix} -\dot{y}(t) \\ \dot{x}(t) \end{pmatrix}!$$

4. Plot $|\dot{\mathbf{r}}(t)|$, $|\ddot{\mathbf{r}}(t)|$, $\ddot{\mathbf{r}}(t)\mathbf{t}(t)$ and $\ddot{\mathbf{r}}(t)\mathbf{n}(t)$ üover a period T!
5. Discuss the temporal behaviour based on the trajectory $\mathbf{r}(t)$! What are the maximum speeds and accelerations, v_{max}, q_{max}?

Fraunhofer FKIE

Exercise 3.2

Consider a sensor in the coordinate origin with constant revisit interval Δt = 5 s measuring the Cartesian position $\mathbf{r}_k = \mathbf{r}(t_k)$ of the objects at times $t_k = k\Delta t$, $k = 0, 1, \ldots$ with Gaussian measurement errors (independent, identically distributed: i.i.d.), covariance matrix $\mathbf{R} = \sigma^2\mathbf{I}$, σ = 50 m. **Mandatory!**

1. Simulate with a random generator measurements $\mathbf{z}_k = (z_k^x, z_k^y)$ according to:

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \sigma \begin{pmatrix} u_k^x \\ u_k^y \end{pmatrix}, \qquad \text{mit} \quad \mathbf{x}_k = \left(\mathbf{r}(t_k)^\top, \dot{\mathbf{r}}(t_k)^\top, \ddot{\mathbf{r}}(t_k)^\top\right)^\top,$$

$$\mathbf{H} = (\mathbf{I}, \mathbf{O}, \mathbf{O}), \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_k^x, u_k^y \sim N(0, 1)$$

   and plot the time series over the trajectory $\mathbf{r}(t)$!

2. Initiate the Kalman recursion by the a priori knowledge:

$$p(\mathbf{x}_0) = \mathcal{N}\left(\mathbf{x}_0;\, \mathbf{x}_{0|0}, \mathbf{P}_{0|0}\right), \quad \mathbf{x}_{0|0} = \begin{pmatrix} \mathbf{z}_0 \\ \mathbf{o} \\ \mathbf{o} \end{pmatrix}, \quad \mathbf{P}_{0|0} = \begin{pmatrix} \mathbf{R} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & v_{\text{max}}^2\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & q_{\text{max}}^2\mathbf{I} \end{pmatrix}, \quad \mathbf{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3. Realize the Kalman recursion as a program and calculate the prediction $\mathbf{x}_{k|k-1}$, $\mathbf{P}_{k|k-1}$ and filtering $\mathbf{x}_{k|k}$, $\mathbf{P}_{k|k}$ für $k = 1, 2, \ldots$. Use the dynamics model:

$$\mathbf{F} = \begin{pmatrix} \mathbf{I} & \Delta t\,\mathbf{I} & \frac{1}{2}\Delta t^2\,\mathbf{I} \\ \mathbf{O} & \mathbf{I} & \Delta t\,\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathrm{e}^{-\Delta t/\theta}\,\mathbf{I} \end{pmatrix}, \quad \mathbf{D} = \Sigma^2(1 - \mathrm{e}^{-2\Delta t/\theta}) \begin{pmatrix} \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}, \quad \Sigma = q_{\text{max}}, \ \theta = 60\text{s}$$

4. Plot measurements, predictions, filtering with corresponding error ellipses (eigen values/vectors!) and compare with the truth! Play with the parameters!

Sensor Fusion: Gain in Localization Accuracy

If a stationary target is observed by N sensors, we naïvely expect an improvement in accuracy $\propto 1/\sqrt{N}$.

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