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- **Solution:** Derive iteration formulae for calculating the pdfs! Develop a mechanism for initiation! By doing so, exploit all background information available! Derive state estimates from the pdfs along with appropriate quality measures!

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How to deal with probability density functions?

- pdf $p(x)$: Extract *probability statements* about the RV x by integration!

- naïvely: *positive* and *normalized* functions ($p(x) \geq 0$, $\int dx\, p(x) = 1$)

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- *conditional pdf* $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on y on RV x?
- *marginal density* $p(x) = \int dy\, p(x,y) = \int dy\, p(x|y)\, p(y)$: Enter y!
- Bayes: $p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int dx\, p(y|x)p(x)}$: $p(x|y) \leftarrow p(y|x)$, $p(x)$!

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- *certain knowledge* on x: $p(x) = \delta(x-y)$ '=' $\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-y)^2}{\sigma^2}}$
- *transformed RV* $y = t[x]$: $p(y) = \int dx\, p(y,x) = \int dx\, p(y|x)\, p_x(x) =$
  $\int dx\, \delta(y - t[x])\, p_x(x) =: [\mathcal{T} p_x](y)$ ($\mathcal{T} : p_x \mapsto p$, "transfer operator")

Characterize an object by *quantitatively describable* properties: object state

Examples:

- object position x on a strait line: $x \in \mathbb{R}$
- kinematic state $\mathbf{x} = (\mathbf{r}^\top, \dot{\mathbf{r}}^\top, \ddot{\mathbf{r}}^\top)^\top$, $\mathbf{x} \in \mathbb{R}^9$

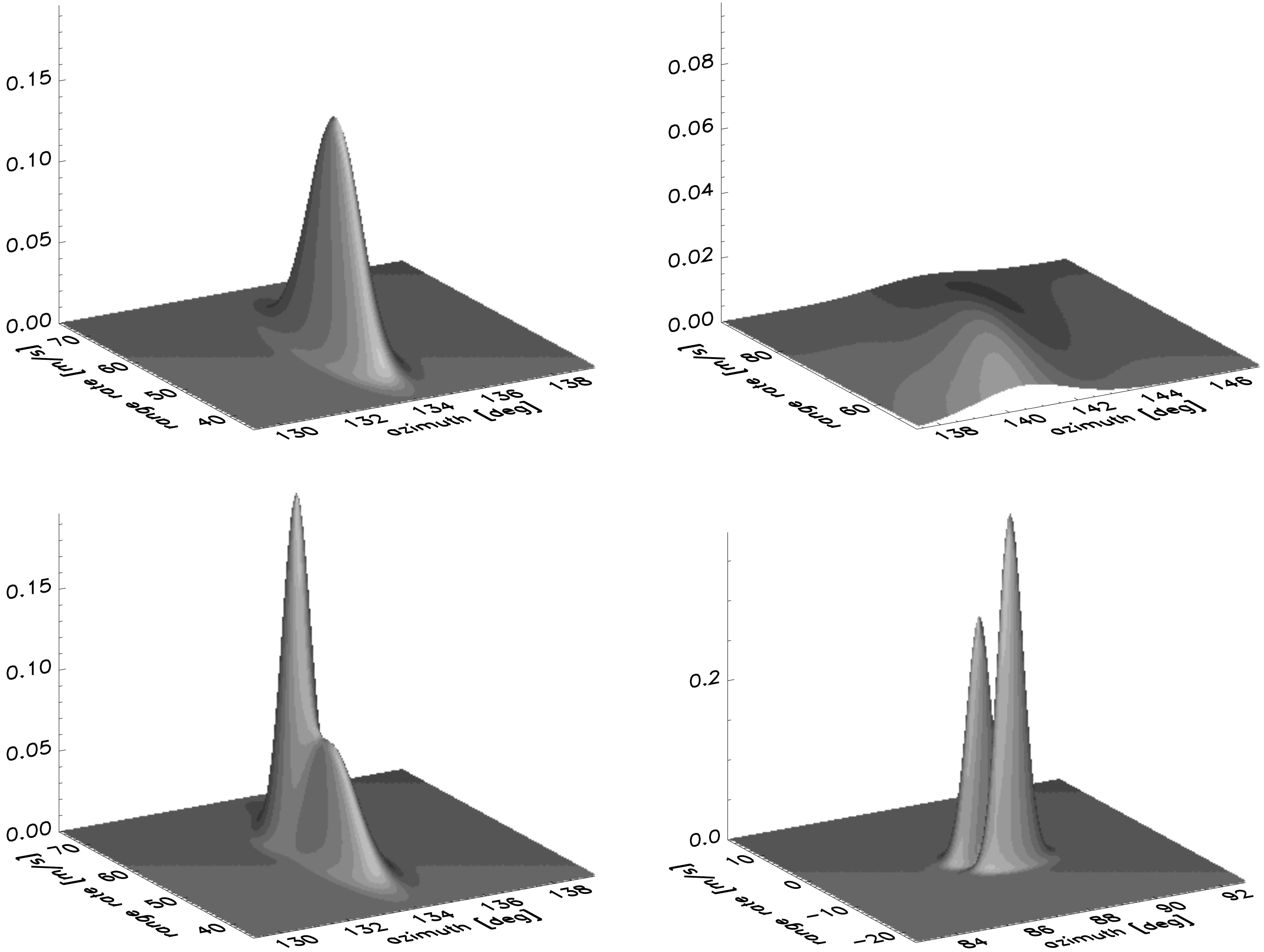
  position $\mathbf{r} = (x, y, z)^\top$, velocity $\dot{\mathbf{r}}$, acceleration $\ddot{\mathbf{r}}$
- joint state of two objects: $\mathbf{x} = (\mathbf{x}_1^\top, \mathbf{x}_2^\top)^\top$
- kinematic state $\mathbf{x}$, object extension $\mathbf{X}$

  z.B. ellipsoid: symmetric, positively definite matrix
- kinematic state $\mathbf{x}$, object class class

  z.B. bird, sailing plane, helicopter, passenger jet, ...

Learn unknown object states from imperfect measurements and describe by functions $p(\mathbf{x})$ imprecise knowledge mathematically precisely!

Interpret unknown object states as *random variables*, x [1D] or $\mathbf{x}$, $\mathbf{X}$ [vector / matrix variate]), characterized by corresponding *probability density functions* (pdf).



The concrete shape of the pdf $p(\mathbf{x})$ contains the full knowledge on $\mathbf{x}$!

Information on a random variable (RV) can be extracted by integration from the corresponding pdf. !

at present: one dimensional case:

How probable is it that $x \in (a, b) \subseteq \mathbb{R}$ ***holds?***

Answer: $P\{x \in (a, b)\} = \int_a^b dx\, p(x) \qquad \Rightarrow \qquad p(x) \geq 0$

in particular: $P\{x \in \mathbb{R}\} = \int_{-\infty}^{\infty} dx\, p(x) = 1$ (normalzation)

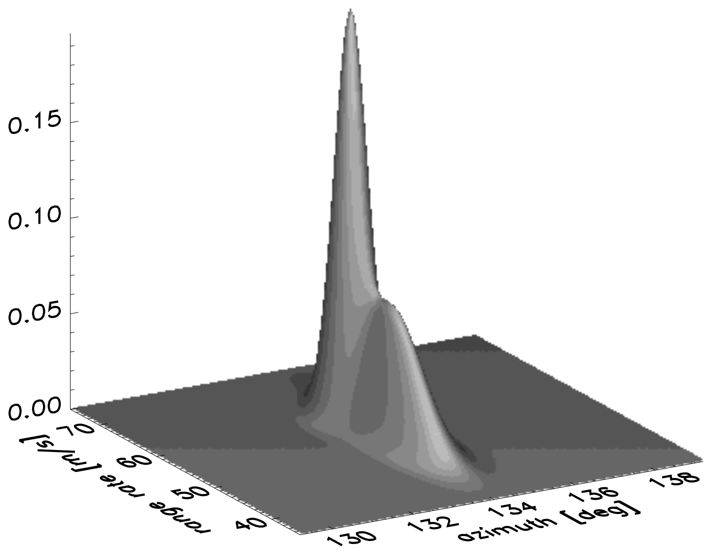
intuitive interpretation: "*the object is somewhere in* $\mathbb{R}$"

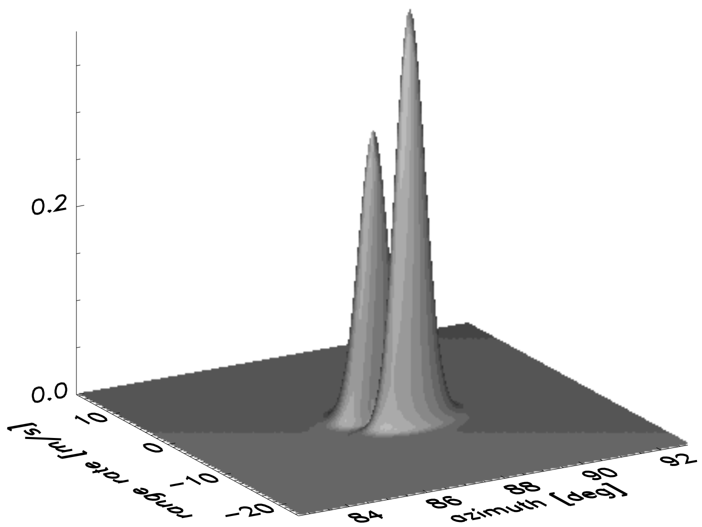
loosely: $p(x)\, dx$ **is probabity for** x **having a value between** x **and** $x + dx$

Recapitulation: How to characterize the properties of a pdf?

specifically: How to associate a single "expected" value to a RV?

The maximum of the pdf is sometimes but not always useful!





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Recapitulation: How to characterize the properties of a pdf?

specifically: How to associate a single "expected" value to a RV?

The maximum of the pdf is sometimes but not always useful! ($\rightarrow$ examples)

instead: Calculate the centroid of the pdf!

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, p(x) = \bar{x}$$ "expectation value"

more generally: Consider functions $g : x \mapsto g(x)$ of the RV x!

$$\mathbb{E}[g(x)] = \int_{-\infty}^{\infty} dx\, g(x)\, p(x),$$ "expectation value of the *observable* g"

Example: Consider the observable $\frac{1}{2}mx^2$ (kinetic energy, x = speed)

Recapitulation: An important observable: the "error" of an estimate

- **Quality:** How useful is an expectation value $\bar{x} = \mathbb{E}[x]$?

Consider special obervables as distance measure:

$$g(x) = |x - \bar{x}| \quad \text{oder} \quad g(x) = (x - \bar{x})^2$$

quadratic measures: computationally more comfortable!

'expected error' of the expectation value $\bar{x}$:

$$\boxed{\mathbb{V}[x] = \mathbb{E}[(x - \bar{x})^2],\ \sigma_x = \sqrt{\mathbb{V}[x]}}$$

variance, standard deviation

Exercise 1.1 Show that $\boxed{\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2}$ holds. **(today!)**

Expectation value of the observable x^2 also called "2nd moment" of the pdf of x.

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Calculate expectation and variance of the uniform density of a RV $x \in \mathbb{R}$ in the intervall $[a, b]$. (**today!**)

Exercise 1.2

$$p(x) = \mathcal{U}(\underbrace{x}_{\textbf{ZV}};\ \underbrace{a, b}_{\textbf{Parameter}}) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \textbf{sonst} \end{cases}$$

!

Pdf correctly normalized? $\displaystyle \int_{-\infty}^{\infty} dx\, \mathcal{U}(x; a, b) = \frac{1}{b-a} \int_a^b dx = 1$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx\, x\, \mathcal{U}(x; a, b) = \frac{b+a}{2}$$

$$\mathbb{V}[x] = \frac{1}{b-a} \int_a^b dx\, x^2 - \mathbb{E}[x]^2 = \frac{1}{12}(b-a)^2$$

Important example: x *normally distributed* over $\mathbb{R}$ (Gauß)

- *wanted:* probabilities concentrated around μ
- quadratic distance: $||x-\mu||^2 = \frac{1}{2}(x-\mu)^2/\sigma^2$ (mathematically convenient!)
- Parameter σ is a measure of the "width" of the pdf: $||\sigma||^2 = \frac{1}{2}$
- for 'large' distances, i.e. $||x-\mu||^2 \gg \frac{1}{2}$, the pdf shall decay quickly.
- simplest approach: $\tilde{p}(x) = \mathrm{e}^{-||x-\mu||^2}$ ($> 0\ \forall x \in \mathbb{R}$, normalization?)
- Normalized for $p(x) = \tilde{p}(x) / \int_{-\infty}^{\infty} dx\, \tilde{p}(x)$!

Formula collection delivers: $\int_{-\infty}^{\infty} dx\, \tilde{p}(x) = \sqrt{2\pi}\sigma$

An admissible pdf with the required properties is obviously given by:

$$\mathcal{N}(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Exercise 1.3

Show for the Gauß density $p(x) = \mathcal{N}(x; \mu, \sigma)$:

$$\boxed{\mathbb{E}[x] = \mu, \ \ \mathbb{V}[x] = \sigma^2}$$

(**today!**)

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} dx \, x \, \mathcal{N}(x; \mu, \sigma) = \mu$$

$$\mathbb{V}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2$$

Use substitution and partial integration!

Use $\int_{-\infty}^{\infty} dx \, \mathrm{e}^{-\frac{1}{2}x^2} = \sqrt{2\pi}$!

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Generalization to multiple random variables!

Vector states: $\mathbf{x} = (x_1, \ldots, x_{n-1}, x_n)^\top$ e.g. : $\mathbf{x} = (\mathbf{r}, \dot{\mathbf{r}})^\top, \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)^\top$

Volume integral: $P\{\mathbf{x} \in V\} = \int_V dx_1 \ldots dx_n \, p(x_1, \ldots, x_n)$

vector variate or scalar expectation values: $\mathbb{E}[g(\mathbf{x})] = \int d\mathbf{x} \, g(\mathbf{x}) \, p(\mathbf{x})$

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Hybrid states: $\mathbf{x}, \mathbf{X}$ (extension, continuous) or $\mathbf{x}$, class (type, discrete)

Expectation values (e.g. $g : \mathbf{x}, \mathbf{X} \mapsto \mathbf{x}$): $\mathbb{E}[g(\mathbf{x}, \mathbf{X})] = \int d\mathbf{x} \, d\mathbf{X} \, g(\mathbf{x}, \mathbf{X}) \, p(\mathbf{x}, \mathbf{X})$

or (e.g. $g : \mathbf{x}, \mathsf{class} \mapsto \mathsf{class}$): $\mathbb{E}[g(\mathbf{x}, \mathsf{class})] = \sum_{\mathsf{class}} \int d\mathbf{x} \, g(\mathbf{x}, \mathsf{class}) \, p(\mathbf{x}, \mathsf{class})$

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Independence: Statements about x not influenced by $y \to \; p(x, y) = p(x)\, p(y)$!

$$P\{x \in X, y \in Y\} = \int_X dx \ldots dx \, p(x) \int_Y dy \ldots dy \, p(y)$$

Example: $\mathbb{E}[\text{class}] = \sum_{\text{class}} \text{class} \, p(\text{class}) \underbrace{\int d\mathbf{x} \, p(\mathbf{x})}_{=1 \text{ normalization!}}$

Some properties of joint densities

Non-negative: $p(x, y) \geq 0$

Normalized: $\int dx\, dy\, p(x, y) = 1$

Relation between $p(x)$, $p(y)$ and $p(x, y)$:

$$p(x) = \int dy\, p(x, y)$$

$$p(y) = \int dx\, p(x, y)$$

$p(x)$ is also called a marginal density of the joint density w.r.t. x.

How does knowledge on a RV y affects knowledge on a RV x?

No impact if x, y are independent of each other. $\rightarrow$ $p(x|y) = p(x)$

Feeling: $p(x, y)$ and $p(y)$ should enter into the definition of $p(x|y)$.

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A first attempt: $$\boxed{p(x|y) = \frac{p(x, y)}{p(y)}}$$

- $\int dx\, p(x|y) = \frac{1}{p(y)} \int dx\, p(x, y) = \frac{p(y)}{p(y)} = 1$ $\quad\rightarrow$ Normalized!
- x, y mutually independent: $p(x|y) = \frac{p(x, y)}{p(y)} = \frac{p(x)\, p(y)}{p(y)} = p(x)$

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- x, y mutually independent: $p(x|y) = \frac{p(x, y)}{p(y)} = \frac{p(x)\, p(y)}{p(y)} = p(x)$

$p(x|y) \geq 0$ is obviously interpretable as a useful pdf that quantitatively describes the notions of statistical “dependency” and “independency”.

conditional probability density function: $\quad p(x|y)$

More Precise Formulation of the BAYESian Approach

Consider a set of measurements $Z_l = \{\mathbf{z}_l^j\}_{j=1}^{m_l}$ of a single or a multiple target state $\mathbf{x}_l$ at time instants t_l, $l = 1, \ldots, k$ and the time series:

$$\mathcal{Z}^k = \{Z_k, m_k, Z_{k-1}, m_{k-1}, \ldots, Z_1, m_1\} = \{Z_k, m_k, \mathcal{Z}^{k-1}\}!$$

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Based on $\mathcal{Z}^k$, what can be learned about the object states $\mathbf{x}_l$ at $t_1, \ldots, t_k, t_{k+1}, \ldots$, i.e. for the past, present, and future?

Evidently the answer is given be calculating the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$!

Fraunhofer FKIE

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multiple sensor measurement fusion: Calculate $p(\mathbf{x}|\mathcal{Z}_1^k, \ldots, \mathcal{Z}_N^k)$!

- communication lines
- common coordinate system: sensor registration

How to calculate conditional pdfs? Use Bayes' Rule!

Because of: $p(x|y)\,p(y) = p(x,y) = p(y,x) = p(y|x)\,p(x)$

we have in particular: $$p(x|y) = \frac{p(y|x)\,p(x)}{p(y)}$$

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We can also write: $$p(y) = \underbrace{\int dx\, p(y,x)}_{\text{marginal pdf}} = \int dx\; \underbrace{p(y|x)\,p(x)}_{\text{def. cond. pdf}}$$

and thus obtain: $$\boxed{p(x|y) = \frac{p(y|x)\,p(x)}{\int dx\, p(y|x)\,p(x)}}$$

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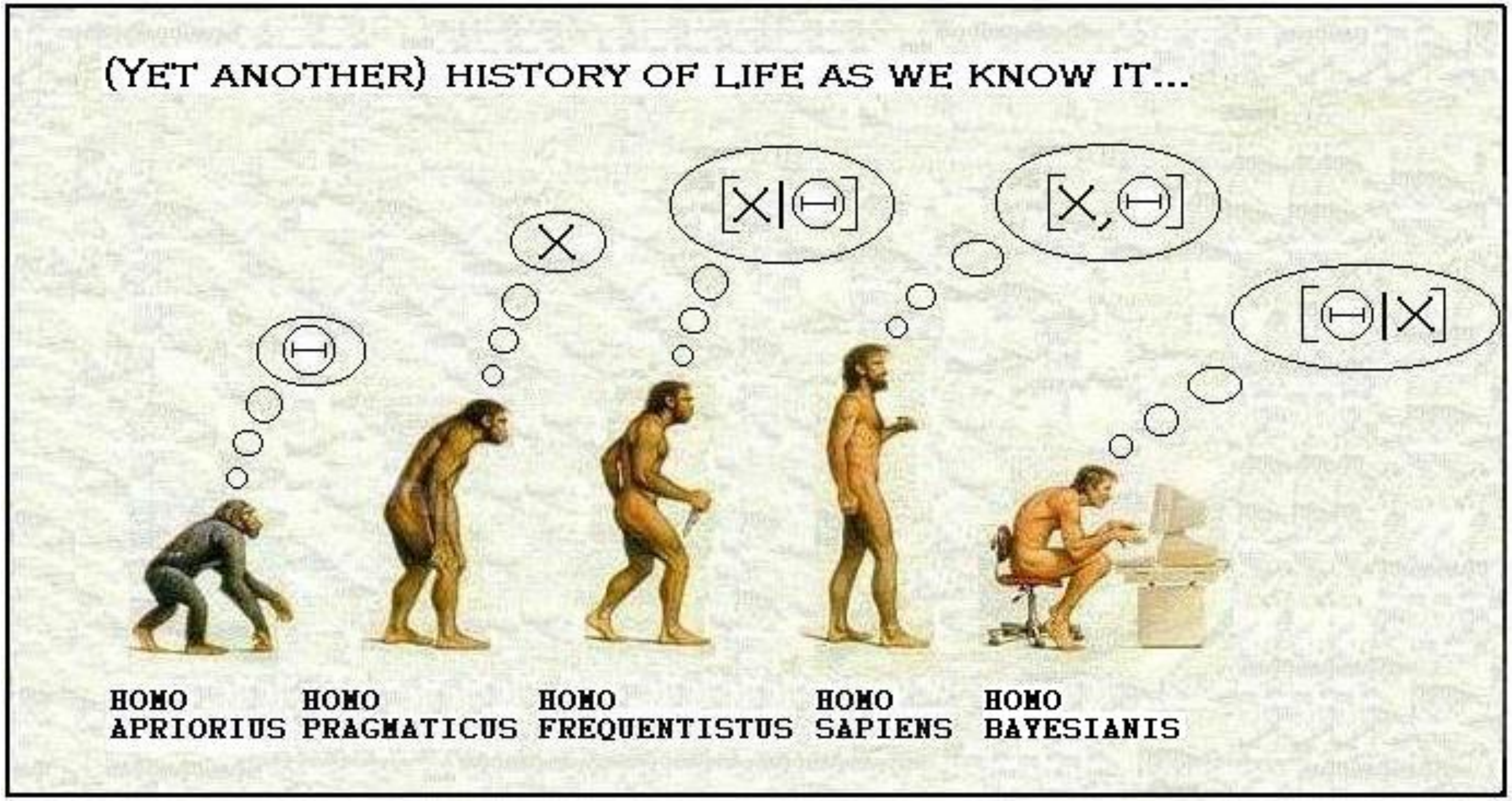
- Who knows $p(y|x)$ and $p(x)$, can calculate how knowledge on y affects knowledge on x.
- Large parts of statistics is just an application of Bayes' rule. (Rev. Thomas Bayes, 18th century, fully understood by Laplace)

REV. T. BAYES

Exercise 2.1 (voluntary)

Provide us a short overview of Rev. Thomas Bayes (1701–1761), perhaps starting with `http://en.wikipedia.org/wiki/Thomas_Bayes`.



Exercise 2.2a (voluntary) Present the "Monty Hall"-problem and solve it using Bayes' rule (see e.g. `http://en.wikipedia.org/wiki/Monty_Hall_problem`).

Exercise 2.2b (voluntary) Read and explain us the following "Spiegel"-article: `http://www.spiegel.de/wissenschaft/mensch/0,1518,708540,00.html`!

Quanten-Bayesianismus.pdf

Magazin | 18.10.2013

TITELTHEMA: QUANTEN-BAYESIANISMUS

Eine neue Quantentheorie

Im Jahr 1926 führte Erwin Schrödinger die Wellenfunktion in die Quantentheorie ein. Wie sie zu verstehen ist, darüber zerbrechen sich die Physiker bis heute die Köpfe. Eine neue Deutung namens QBismus geht davon aus, dass die Wellenfunktion nur die subjektive Erwartungshaltung des quantenmechanischen Beobachters wiedergibt.

HANS CHRISTIAN VON BAEYER

Die Quantenmechanik erklärt das Verhalten der Materie präzise – von subatomaren bis zu astronomischen Größenordnungen. Sie ist die erfolgreichste physikalische Theorie und zugleich die seltsamste: In der Quantenwelt scheinen sich Teilchen an zwei Orten zugleich aufzuhalten; Information pflanzt sich vermeintlich mit Überlichtgeschwindigkeit fort, und Katzen können gleichzeitig tot und lebendig sein. Seit nunmehr 90 Jahren plagen sich Wissenschaftler ohne rechten Erfolg mit solchen Paradoxien herum. Während die Erkenntnisse der Evolutionstheorie und Kosmologie längst fest zu unserem Weltbild gehören, gilt die Quantentheorie sogar vielen Physikern als bizarre Abnormität – eine zwar fast magisch wirksame, aber im Grund unerklärliche Gebrauchsanleitung zum Bau technischer Geräte wie Laser, Transistoren oder

AUF EINEN BLICK

Rein subjektive Beschreibung?

1. Die Quantenmechanik steckt trotz all ihrer Erfolge voller Paradoxien. Ein neues Modell namens Quanten-Bayesianismus – kurz QBismus – kombiniert Quanten- und Wahrscheinlichkeitstheorie, um die Widersprüche zu entschärfen.

2. Der QBismus interpretiert die Quelle aller Quantenparadoxien – die Wellenfunktion – auf neue Weise. Physiker berechnen mit ihr die Wahrscheinlichkeit, dass ein Teilchen eine bestimmte

Spektrum der Wissenschaft, November 2013

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MEISTGELESEN

1. KOSMOLOGIE | Schlechte Nachrichten für Dunkle-Materie-Jäger
2. RAUCHEN | Forschung im Nebel der Interessen
3. ÖKOSYSTEME | Der Wert der Zweiten

http://www.spektrum.de/alias/titelthema-quanten-bayesianismus/eine-neue-quantentheorie/1201696[06.11.2013 11:05:36]

How to calculate the pdf $p(\mathbf{x}_l|\mathcal{Z}^k)$?

Consider at first the present time: $l = k$.

an observation:

Bayes' rule: $p(\mathbf{x}_k|\mathcal{Z}^k) = p(\mathbf{x}_k|Z_k, m_k, \mathcal{Z}^{k-1})$

$$= \frac{p(Z_k, m_k|\mathbf{x}_k, \mathcal{Z}^{k-1})\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, p(Z_k, m_k|\mathbf{x}_k, \mathcal{Z}^{k-1})\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

Fraunhofer FKIE

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$$p(x) = \mathcal{N}(x;\, \mathbf{x},\, \mathbf{P}) = \frac{1}{\sqrt{|2\pi\mathbf{P}|}}\, \mathrm{e}^{-\frac{1}{2}(x-\mathbf{x})^\top \mathbf{P}^{-1}(x-\mathbf{x})}$$

Exercise 2.3 Show: $\int dx\, \mathrm{e}^{-q(x)} = \sqrt{|2\pi\mathbf{P}|}$, $\mathbb{E}[x] = \mathbf{x}$, $\mathbb{E}[(x - \mathbf{x})(x - \mathbf{x})^\top] = \mathbf{P}$

Trick: Symmetric, positively definite matrices can be diaginalized by an orthogonal coordinate transform.

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– GAUSS*ian Mixtures:* $p(x) = \sum_i p_i\, \mathcal{N}(x;\, \mathbf{x}_i,\, \mathbf{P}_i)$ (weighted sums)

A (very!) useful product formula for GAUSSians

Exercise 2.4 (voluntary) Show:

$$\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\, \mathbf{y},\, \mathbf{P}\big) = \underbrace{\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{y},\, \mathbf{S}\big)}_{\text{independent of } \mathbf{x}} \times \begin{cases} \mathcal{N}\big(\mathbf{x};\, \mathbf{y} + \mathbf{W}\boldsymbol{\nu},\, \mathbf{P} - \mathbf{W}\mathbf{S}\mathbf{W}^\top\big) \\ \mathcal{N}\big(\mathbf{x};\, \mathbf{Q}^{-1}(\mathbf{P}^{-1}\mathbf{x} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\, \mathbf{Q}\big) \end{cases}$$

$$\boldsymbol{\nu} = \mathbf{z} - \mathbf{H}\mathbf{y}, \quad \mathbf{S} = \mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{R}, \quad \mathbf{W} = \mathbf{P}\mathbf{H}^\top\mathbf{S}^{-1}, \quad \mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}.$$

Sketch of the proof:

- Interpret $\mathcal{N}\big(\mathbf{z};\, \mathbf{H}\mathbf{x},\, \mathbf{R}\big)\,\mathcal{N}\big(\mathbf{x};\, \mathbf{y},\, \mathbf{P}\big)$ as a joint pdf $p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{z}, \mathbf{x})$.
- Show that $p(\mathbf{z}, \mathbf{x})$ is a GAUSSian: $p(\mathbf{z}, \mathbf{x}) = \mathcal{N}\Big(\big(\begin{smallmatrix} \mathbf{z} \\ \mathbf{x} \end{smallmatrix}\big);\, \big(\begin{smallmatrix} \mathbf{H}\mathbf{y} \\ \mathbf{y} \end{smallmatrix}\big),\, \big(\begin{smallmatrix} \mathbf{S} & \mathbf{H}\mathbf{P} \\ \mathbf{P}\mathbf{H}^\top & \mathbf{P} \end{smallmatrix}\big)\Big)$.
- Calculate from $p(\mathbf{z}, \mathbf{x})$ the marginal and conditional pdfs $p(\mathbf{z})$ and $p(\mathbf{x}|\mathbf{z})$.
- From $p(\mathbf{z}, \mathbf{x}) = p(\mathbf{z}|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) = p(\mathbf{x}, \mathbf{z})$ we obtain the result.

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often: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = p(\mathbf{x}_k|\mathbf{x}_{k-1})$ (MARKOV)

sometimes: $p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}}_{\text{deterministic}}\mathbf{x}_{k-1},\, \underbrace{\mathbf{D}_{k|k-1}}_{\text{random}}\big)$ (linear GAUSS-MARKOV)

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sometimes: $\ell(\mathbf{z}_k; \mathbf{x}_k) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$ (1 target, 1 measurement)

iteration formula:
$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, \int d\mathbf{x}_{k-1}\, p(\mathbf{x}_k|\mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1}|\mathcal{Z}^{k-1})}$$

GAUSSian transition pdf: $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathcal{Z}^{k-1}) = \mathcal{N}(\mathbf{x}_k;\, \mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\, \mathbf{D}_{k|k-1})$

with: $\underbrace{\mathbf{F}_{k|k-1} \text{ (evolution matrix)}}_{\text{describes deterministic motion}}$, $\underbrace{\mathbf{D}_{k|k-1} \text{ (dynamics covariance matrix)}}_{\text{models random maneuvers}}$

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$$
\begin{aligned}
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&= \mathcal{N}\big(\mathbf{x}_k;\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}}_{=:\mathbf{x}_{k|k-1}},\, \underbrace{\mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}^{\top} + \mathbf{D}_{k|k-1}}_{=:\mathbf{P}_{k|k-1}}\big) \\
&\quad \times \underbrace{\int d\mathbf{x}_{k-1}\, \mathcal{N}\big(\mathbf{x}_{k-1};\, \ldots,\, \ldots\big)}_{=1 \quad \text{(normalization!)}} \qquad \text{(exploit product formula!)}
\end{aligned}
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Fraunhofer FKIE

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$$= \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$$

Fraunhofer FKIE

Bayesian filtering update based on predictions (KALMAN filter)

$$p(\mathbf{x}_k|\mathcal{Z}^k) = \frac{\ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}{\int d\mathbf{x}_k\, \ell(\mathbf{x}_k; Z_k, m_k)\, p(\mathbf{x}_k|\mathcal{Z}^{k-1})}$$

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Consider as a simple example $\ell(\mathbf{x}_k; Z_k, m_k) = \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{H}_k\mathbf{x}_k,\, \mathbf{R}_k\big)$!

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$$= \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k}\big)$$ (due to the product formula)

with: $\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{x}_{k|k}(\mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}),\qquad \mathbf{W}_{k|k} = \mathbf{P}_{k|k-1}\mathbf{H}_k^\top\mathbf{S}_{k|k}$ (KALMAN update)

$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k}\mathbf{W}_{k|k-1},\qquad \mathbf{S}_{k|k} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k + \mathbf{R}_k.$

Fraunhofer FKIE

a first remark on initiation: $p(\mathbf{x}_0|\mathcal{Z}^0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, $\mathbf{P}_{0|0}$ 'large'

$$\mathbf{x}_{0|0} = \begin{pmatrix} \mathbf{r}_{0|0} \\ \dot{\mathbf{r}}_{0|0} \\ \ddot{\mathbf{r}}_{0|0} \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{P}_{0|0} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (v_{max})^2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (q_{max})^2\mathbf{1} \end{pmatrix}$$

position information: first measurement $\mathbf{z}_0$, ignorance = measurement error $\mathbf{R}$!

ignorance on velocity: sphere with radius v_{max} around zero
(= no information on direction, but on 'limits')

ignorance on acceleration: sphere with radius q_{max} around zero

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}\big(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0}\big)$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

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prediction: $\mathcal{N}\big(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}\big) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}\big(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}\big)$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

Kalman filter: linear GAUSSian likelihood/dynamics, $\mathbf{x}_k = (\mathbf{r}_k^\top, \dot{\mathbf{r}}_k^\top, \ddot{\mathbf{r}}_k^\top)^\top$, $\mathcal{Z}^k = \{\mathbf{z}_k, \mathcal{Z}^{k-1}\}$

initiation: $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0;\, \mathbf{x}_{0|0},\, \mathbf{P}_{0|0})$, initial ignorance: $\mathbf{P}_{0|0}$ 'large'

prediction:

$$\mathcal{N}(\mathbf{x}_{k-1};\, \mathbf{x}_{k-1|k-1},\, \mathbf{P}_{k-1|k-1}) \xrightarrow[\mathbf{F}_{k|k-1},\, \mathbf{D}_{k|k-1}]{\text{dynamics model}} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1})$$

$$\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k|k-1}{}^\top + \mathbf{D}_{k|k-1}$$

filtering:

$$\mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k-1},\, \mathbf{P}_{k|k-1}) \xrightarrow[\text{sensor model: } \mathbf{H}_k,\, \mathbf{R}_k]{\text{current measurement } \mathbf{z}_k} \mathcal{N}(\mathbf{x}_k;\, \mathbf{x}_{k|k},\, \mathbf{P}_{k|k})$$

$$\mathbf{x}_{k|k} = \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\boldsymbol{\nu}_{k|k-1}, \qquad \boldsymbol{\nu}_{k|k-1} = \mathbf{z}_k - \mathbf{H}_k\mathbf{x}_{k|k-1}$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1}\mathbf{S}_{k|k-1}\mathbf{W}_{k|k-1}{}^\top, \qquad \mathbf{S}_{k|k-1} = \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top + \mathbf{R}_k$$

$$\mathbf{W}_{k|k-1} = \mathbf{P}_{k|k-1}\mathbf{H}_k{}^\top\mathbf{S}_{k|k-1}{}^{-1}$$ 'KALMAN gain matrix'