Introduction to Sensor Data Fusion
Methods and Applications

• Last lecture: Why Sensor Data Fusion?
  – Motivation, general context
  – Discussion of examples

• Today: Steep climb to a first algorithm.

• oral examination: 6 credit points after the end of the semester
• prerequisite: participate in the excercises, explain a good program
• job opportunities as research assistant in ongoing projects, practicum
• subsequently: bachelor at Fraunhofer FKIE, master / PhD possible
• slides/script: email to wolfgang.koch@fkie.fraunhofer.de, download
Sensor & Information Fusion: Basic Task

information sources: defined by operational requirements
Sensor & Information Fusion: Basic Task

**information to be fused**: imprecise, incomplete, ambiguous, unresolved, false, deceptive, hard to formalize, contradictory...
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information sources: defined by operational requirements
Exercise 2.1

Consider an object that moves in two dimensions on the trajectory:

\[
\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} \sin(\omega t) \\ \sin(2\omega t) \end{pmatrix}
\]

with \( A = \frac{v^2}{q} \), \( \omega = \frac{q}{2v} \)

and speed and acceleration parameters: \( v = 300 \, \text{m/s}, \quad q = 9 \, \text{m/s}^2 \)

1. Plot the trajectory. Why is it periodical? What is its period \( T = T(v, q) \)?

2. Show for the velocity and acceleration vector:

\[
\mathbf{\dot{r}}(t) = v \begin{pmatrix} \cos(\omega t)/2 \\ \frac{\sin(\omega t)}{\sin(2\omega t)} \end{pmatrix}, \quad \mathbf{\ddot{r}}(t) = -q \begin{pmatrix} \sin(\omega t)/4 \\ \sin(2\omega t) \end{pmatrix}
\]

3. Calculate for each instance of time \( t \) the tangential and normal vectors in \( \mathbf{r}(t) \):

\[
\mathbf{t}(t) = \frac{1}{|\mathbf{r}(t)|} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}, \quad \mathbf{n}(t) = \frac{1}{|\mathbf{r}(t)|} \begin{pmatrix} -\dot{y}(t) \\ \dot{x}(t) \end{pmatrix}
\]

4. Plot \( |\mathbf{\dot{r}}(t)|, |\mathbf{\ddot{r}}(t)|, \mathbf{\dot{r}}(t)\mathbf{t}(t) \) and \( \mathbf{\dot{r}}(t)\mathbf{n}(t) \) over a period \( T \)!

5. Discuss the temporal behaviour based on the trajectory \( \mathbf{r}(t) \)!

6. What are the maximum speeds and accelerations, \( v_{\text{max}}, q_{\text{max}} \)?
Probability densities functions (pdf) represent imprecise knowledge on the ‘state’ \( x_{k-1} \) based on imprecise measurements \( Z_{k-1} \).
Characterize an object by *quantitatively describable* properties: object state

- object position $x$ on a straight line: $x \in \mathbb{R}$
- kinematic state $x = (r^T, \dot{r}^T, \ddot{r}^T)^T$, $x \in \mathbb{R}^9$
  position $r = (x, y, z)^T$, velocity $\dot{r}$, acceleration $\ddot{r}$
- joint state of two objects: $x = (x_1^T, x_2^T)^T$
- kinematic state $x$, object extension $X$
  z.B. ellipsoid: symmetric, positively definite matrix
- kinematic state $x$, object class *class*
  z.B. bird, sailing plane, helicopter, passenger jet, ...

Learn unknown object states from imperfect measurements and describe by functions $p(x)$ imprecise knowledge mathematically precisely!
How to deal with probability density functions?

- pdf \( p(x) \): Extract *probability statements* about the RV \( x \) by integration!

- na"ively: *positive* and *normalized* functions \((p(x) \geq 0, \int dx \, p(x) = 1)\)
Exploit imprecise knowledge on the **dynamical behavior** of the object.

\[
p(x_k | Z_{k-1}) = \int dx_{k-1} \quad p(x_k | x_{k-1}) \quad p(x_{k-1} | Z_{k-1}).
\]

- **Prediction**: $t_{k-1}$
- **Prädiktion**: $t_k$
How to deal with probability density functions?

- pdf $p(x)$: Extract \textit{probability statements} about the RV $x$ by integration!

- naïvely: positive and normalized functions ($p(x) \geq 0, \int dx \ p(x) = 1$)

- \textit{conditional pdf} $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on $y$ on RV $x$?

- \textit{marginal density} $p(x) = \int dy \ p(x,y) = \int dy \ p(x|y) \ p(y)$: Enter $y$!

$$p(x_k|Z^{k-1}) = \int dx_{k-1} \ p(x_k, x_{k-1}|Z^{k-1})$$

$$= \int dx_{k-1} \ p(x_k|x_{k-1}, Z^{k-1}) \ p(x_{k-1}|Z^{k-1})$$

$$= \int dx_{k-1} \ p(x_k|x_{k-1}) \ p(x_{k-1}|Z^{k-1})$$
missing sensor detection: ‘data processing’ = prediction
(not always: exploitation of ‘negative’ sensor evidence)
missing sensor information: increasing knowledge dissipation
sensor information on the kinematical object state
**Bayes’ formula:**

\[
p(x_{k+1}|Z^{k+1}) = \frac{p(z_{k+1}|x_{k+1}) \ p(x_{k+1}|Z^k)}{\int d x_{k+1} \ p(z_{k+1}|x_{k+1}) \ p(x_{k+1}|Z^k)}
\]

- pdf: \( t_{k-1} \)
- likelihood (Sensormodell)
- pdf: \( t_k \)
- Prädiktion: \( t_{k+1} \)
How to deal with probability density functions?

- pdf $p(x)$: Extract *probability statements* about the RV $x$ by integration!

- naively: *positive* and *normalized* functions ($p(x) \geq 0$, $\int dx \ p(x) = 1$)

- *conditional pdf* $p(x|y) = \frac{p(x,y)}{p(y)}$: Impact of information on $y$ on RV $x$?

- *marginal density* $p(x) = \int dy \ p(x,y) = \int dy \ p(x|y) \ p(y)$: Enter $y$!

- Bayes: $p(x|y) = \frac{p(y|x) \ p(x)}{p(y)} = \frac{p(y|x) \ p(x)}{\int dx \ p(y|x) \ p(x)}$: $p(x|y) \leftarrow p(y|x), \ p(x)$!

\[
p(x|y) \ p(y) = p(x, y) = p(y, z) = p(y|x) \ p(x)
\]
filtering = sensor data processing
Target or Object Tracking: Basic Idea

Iterative updating of conditional probability densities!

- **prediction:** $p(x_{k-1} | Z^{k-1}) \xrightarrow{\text{dynamics model}} p(x_k | Z^{k-1})$
- **filtering:** $p(x_k | Z^{k-1}) \xrightarrow{\text{sensor model}} p(x_k | Z^k)$
- **retrodiction:** $p(x_{l-1} | Z^k) \xleftarrow{\text{filtering output}} p(x_l | Z^k)$

- Kinematic target state $x_k$ at time $t_k$, accumulated sensor data $Z^k$
- A priori knowledge: target dynamics models, sensor model
Exploit imprecise knowledge on the **dynamical behavior** of the object.

\[
p(x_k|Z^{k-1}) = \int dx_{k-1} \ p(x_k|x_{k-1}) \ p(x_{k-1}|Z^{k-1}).
\]
The Multivariate Gaussian Pdf

– **wanted:** probabilities ‘concentrated’ around a center \( x \)

– **quadratic distance:** \( q(x) = \frac{1}{2} (x - x)P^{-1}(x - x)^\top \)

\( q(x) \) defines an ellipsoid around \( x \), its volume and orientation being determined by a matrix \( P \) (symmetric: \( P^\top = P \), positively definite: all eigenvalues \( > 0 \)).

– **first attempt:** \( p(x) = e^{-q(x)} / \int dx \; e^{-q(x)} \) (normalized!)

\[
p(x) = \mathcal{N}(x; \; x, \; P) = \frac{1}{\sqrt{|2\pi P|}} e^{-\frac{1}{2} (x - x)^\top P^{-1} (x - x)}
\]

– **Gaussian Mixtures:** \( p(x) = \sum_i p_i \; \mathcal{N}(x; \; x_i, \; P_i) \) (weighted sums)
pdf: $t_{k-1}$

Prädiktion: $t_k$

Exploit imprecise knowledge on the dynamical behavior of the object.

$$p(x_k | Z^{k-1}) = \int dx_{k-1} \; \mathcal{N}(x_k; Fx_{k-1}, D) \; \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) .$$
A Useful Product Formula for GAUSSians

\[
\mathcal{N}(z; Fx, D) \mathcal{N}(x; y, P) = \mathcal{N}(z; Fy, S) \mathcal{N}(x; y + W\nu, P - WSW^T) \]

\[
\nu = z - Fy, \quad S = FPF^T + D, \quad W = PF^T S^{-1}.
\]
Kalman filter: $x_k = (r_k^T, \dot{r}_k^T)^T$, $\mathcal{Z}^k = \{z_k, \mathcal{Z}^{k-1}\}$

**initiation:**
\[
p(x_0) = \mathcal{N}(x_0; x_{0|0}, P_{0|0}), \quad \text{initial ignorance: } P_{0|0} \text{ 'large'}
\]

**prediction:**
\[
\mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \xrightarrow{\text{dynamics model}} \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1})
\]
\[
x_{k|k-1} = F_{k|k-1} x_{k-1|k-1}
\]
\[
P_{k|k-1} = F_{k|k-1} P_{k-1|k-1} F_{k|k-1}^T + D_{k|k-1}
\]
pdf: $t_{k-1}$

$t_k$: \textit{kein} plot

missing sensor detection: ‘data processing’ = prediction

(not always: exploitation of ‘negative’ sensor evidence)
missing sensor information: increasing **knowledge dissipation**
sensor information on the kinematical object state
**Bayes’ formula:**

\[
p(x_{k+1} \mid Z^{k+1}) = \frac{p(z_{k+1} \mid x_{k+1}) p(x_{k+1} \mid Z^k)}{\int dx_{k+1} p(z_{k+1} \mid x_{k+1}) p(x_{k+1} \mid Z^k)}
\]
Bayes’ formula:

\[
p(x_{k+1} | Z^{k+1}) = \frac{\mathcal{N}(z_{k+1}; Hx_{k+1}, R) \mathcal{N}(x_{k+1}; x_{k+1|k}, P_{k+1|k})}{\int d x_{k+1} \mathcal{N}(z_{k+1}; Hx_{k+1}, R) \mathcal{N}(x_{k+1}; x_{k+1|k}, P_{k+1|k})}
\]
A Useful Product Formula for GAUSSians

\[
\mathcal{N}(z; Hx, R) \mathcal{N}(x; y, P) = \underbrace{\mathcal{N}(z; Hy, S)}_{\text{independent of } x} \mathcal{N}(x; y + W\nu, P - WSW^\top)
\]

\[
\nu = z - Hy, \quad S = HPH^\top + R, \quad W = PH^\top S^{-1}.
\]
Kalman filter: 
\[ x_k = (r_k^T, \dot{r}_k^T)^T, \mathcal{Z}^k = \{z_k, \mathcal{Z}^{k-1}\} \]

**initiation:**
\[ p(x_0) = \mathcal{N}(x_0; x_{0|0}, P_{0|0}) \]
initial ignorance: \( P_{0|0} \) ‘large’

**prediction:**
\[ \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \xrightarrow{\text{dynamics model}} \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \]
\[ x_{k|k-1} = F_{k|k-1} x_{k-1|k-1} \]
\[ P_{k|k-1} = F_{k|k-1} P_{k-1|k-1} F_{k|k-1}^T + D_{k|k-1} \]

**filtering:**
\[ \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \xrightarrow{\text{current measurement } z_k} \mathcal{N}(x_k; x_{k|k}, P_{k|k}) \]
\[ x_{k|k} = x_{k|k-1} + W_{k|k-1} \nu_{k|k-1}, \quad \nu_{k|k-1} = z_k - H_k x_{k|k-1} \]
\[ P_{k|k} = P_{k|k-1} - W_{k|k-1} S_{k|k-1} W_{k|k-1}^T, \quad S_{k|k-1} = H_k P_{k|k-1} H_k^T + R_k \]
\[ W_{k|k-1} = P_{k|k-1} H_k^T S_{k|k-1}^{-1} \]
‘KALMAN gain matrix’
filtering = sensor data processing
pdf: $t_{k-1}$

pdf: $t_k$

$t_{k+1}$: drei plots

ambiguities by false plots: 1 + 3 data interpretation hypotheses

('detection probability', false alarm statistics)
The Multivariate Gaussian Pdf

– wanted: probabilities ‘concentrated’ around a center $x$

– quadratic distance: $q(x) = \frac{1}{2} (x - x)^\top P^{-1} (x - x)$

$q(x)$ defines an ellipsoid around $x$, its volume and orientation being determined by a matrix $P$ (symmetric: $P^\top = P$, positively definite: all eigenvalues $> 0$).

– first attempt: $p(x) = e^{-q(x)} / \int dx \ e^{-q(x)}$ (normalized!)

$$p(x) = \mathcal{N}(x; \ x, \ P) = \frac{1}{\sqrt{|2\pi P|}} e^{-\frac{1}{2} (x-x)^\top P^{-1} (x-x)}$$

– Gaussian Mixtures: $p(x) = \sum_i p_i \ \mathcal{N}(x; \ x_i, \ P_i)$ (weighted sums)
pdf: $t_{k-1}$

t_{k+1}: drei plots

$p(Z_k, m_k|x_k) = \text{const.} \left( (1 - P_D) \rho_F + P_D \sum_{j=1}^{m_k} \mathcal{N}(z^j_k; Hx_k, R^j_k) \right)$
Multimodal pdfs reflect ambiguities inherent in the data.
temporal propagation: dissipation of the probability densities
association tasks: sensor data $\leftrightarrow$ interpretation hypotheses
Bayes: \[ p(x_{k+2} \mid \mathcal{Z}^{k+2}) = \frac{p(z_{k+2} \mid x_{k+2}) p(x_{k+2} \mid \mathcal{Z}^{k+1})}{\int dx_{k+2} p(z_{k+2} \mid x_{k+2}) p(x_{k+2} \mid \mathcal{Z}^{k+1})} \]
pdf: $t_{k-1}$

pdf: $t_k$    pdf: $t_{k+2}$

pdf: $t_{k+1}$

in particular: **re-calculation** of the hypothesis weights
How does new knowledge affect the knowledge in the past of a past state?
‘retrodiction’: a retrospective analysis of the past
optimal information processing at present and for the past
Multiple Hypothesis Tracking: Basic Idea

*Iterative updating of conditional probability densities!*

- **kinematic target state** $x_k$ at time $t_k$, **accumulated sensor data** $Z^k$
- **a priori knowledge**: target dynamics models, sensor model, road maps

- **prediction**: $p(x_{k-1} | Z^{k-1})$ \(\xrightarrow{\text{dynamics model}}\) $p(x_k | Z^{k-1})$

- **filtering**: $p(x_k | Z^{k-1})$ \(\xrightarrow{\text{sensor model}}\) $p(x_k | Z^k)$

- **retrodiction**: $p(x_{l-1} | Z^k)$ \(\xleftarrow{\text{filtering output}}\) $p(x_l | Z^k)$

- **finite mixture**: inherent ambiguity (data, model, road network)
- **optimal estimators**: e.g. minimum mean squared error (MMSE)
- **initiation of pdf iteration**: multiple hypothesis track extraction