Recapitulation: A popular model for object evolutions

**Piecewise Constant White Acceleration Model**

\[ p(x_k|x_{k-1}) = \mathcal{N}(x_k; F_{k|k-1}x_{k-1}, D_{k|k-1}) \]

Consider state vectors: \( x_k = (r_k^\top, \dot{r}_k^\top)^\top \) (position, velocity)

\[
F_{k|k-1} = \begin{pmatrix} I & \Delta T_k I \\ O & I \end{pmatrix}, \quad D_{k|k-1} = \sum_k^2 \begin{pmatrix} \frac{1}{4} \Delta T_k^4 I & \frac{1}{2} \Delta T_k^3 I \\ \frac{1}{2} \Delta T_k^3 I & \Delta T_k^2 I \end{pmatrix}
\]

Consider state vectors \( x_k = (r_k^\top, \dot{r}_k^\top, \ddot{r}_k^\top)^\top \) (position, velocity, acceleration)

\[
F_{k|k-1} = \begin{pmatrix} I & \Delta T_k I & \frac{1}{2} \Delta T_k^2 I \\ O & I & \Delta T_k I \\ O & O & I \end{pmatrix}, \quad D_{k|k-1} = \sum_k^2 \begin{pmatrix} \frac{1}{4} \Delta T_k^4 I & \frac{1}{2} \Delta T_k^3 I & \frac{1}{2} \Delta T_k^2 I \\ \frac{1}{2} \Delta T_k^3 I & \Delta T_k^2 I & \Delta T_k I \\ \frac{1}{2} \Delta T_k^2 I & \Delta T_k I & I \end{pmatrix}
\]

with \( \Delta T_k = t_k - t_{k-1} \). Reasonable choice: \( \frac{1}{2} v_{\text{max}}/v_{\text{max}} \leq \sum_k \leq v_{\text{max}}/q_{\text{max}} \)
Another, rather realistic model (van Keuk):

\[
\mathbf{F}_{k|k-1} = \begin{pmatrix}
\mathbf{I} & (t_k - t_{k-1}) \mathbf{I} & \frac{1}{2}(t_k - t_{k-1})^2 \mathbf{I} \\
\mathbf{O} & \mathbf{I} & (t_k - t_{k-1}) \mathbf{I} \\
\mathbf{O} & \mathbf{O} & e^{-(t_k - t_{k-1})/\theta} \mathbf{I}
\end{pmatrix}, \quad \mathbf{I} = \text{diag}[1, 1, 1]
\]

\[
\mathbf{D}_{k|k-1} = \Sigma^2 \left(1 - e^{-2(t_k - t_{k-1})/\theta}\right) \begin{pmatrix}
\mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & \mathbf{O} & \mathbf{I}
\end{pmatrix}, \quad \mathbf{O} = \text{diag}[0, 0, 0]
\]

*maneuver correlation time* \(\theta\) (z.B. 60 s), *limiting acceleration* \(\Sigma\) (z.B. 2 \(\text{g}\))

There are many different evolution models adapted to particular problems!

Show for the acceleration process:

**Exercise 5.1 (voluntary!)**

\[
\mathbb{E}[\ddot{\mathbf{r}}_k] = 0, \quad \mathbb{E}[\dot{\mathbf{r}}_k \dot{\mathbf{r}}_l^\top] = \Sigma^2 e^{-(t_k - t_l)/\theta} \mathbf{I}, \quad l \leq k
\]

\(\mathbb{E}[\dot{\mathbf{r}}_k \dot{\mathbf{r}}_l^\top]\) is called ‘auto correlation function’.
Idealized measurement process

- **linear measurement equation:**
  \[ z_k = H_k x_k + u_k, \quad p(u_k) = \mathcal{N}(u_k; 0, R_k) \]
  - to be measured: *linear* functions of the object state
  - measurement error: biasfree, Gaussian distrib.
    independent for different \( t_k \)
  - \( y_k = z_k - H_k x_k \) has the pdf: \( p(y_k) = p(u_k) \)

- **Approach for the requested pdf (‘likelihood fkt.):**
  \[ p(z_k | x_k) = \mathcal{N}(z_k; H_k x_k, R_k) \]

- **Example: position measurement**
  \[ H_k = (I, O, O), \quad H_k x_k = r_k \]
  \( R_k \): measurement error covariance matrix
  possibly depending on the sensor-to-target geometry
a first remark on initiation: \[ p(x_0|\mathcal{Z}^0) = \mathcal{N}(x_0; x_{0|0}, P_{0|0}), \quad P_{0|0} \text{ 'large'} \]

\[
x_{0|0} = \begin{pmatrix} r_{0|0} \\ \dot{r}_{0|0} \\ \ddot{r}_{0|0} \end{pmatrix} = \begin{pmatrix} z_0 \\ 0 \\ 0 \end{pmatrix}, \quad P_{0|0} = \begin{pmatrix} R & 0 & 0 \\ 0 & (v_{max})^2 & 1 \\ 0 & 0 & (q_{max})^2 \end{pmatrix}
\]

position information: first measurement \( z_0 \), ignorance = measurement error \( R \)!

ignorance on velocity: sphere with radius \( v_{max} \) around zero
(= no information on direction, but on ‘limits’)

ignorance on acceleration: sphere with radius \( q_{max} \) around zero
Kalman filter: \( x_k = (r_k^T, \dot{r}_k^T)^T, \mathcal{Z}_k = \{z_k, \mathcal{Z}_{k-1}\} \)

**initiation:**  \[ p(x_0) = \mathcal{N}(x_0; x_{0|0}, P_{0|0}) \]

**initial ignorance:**  \( P_{0|0} \) ‘large’

**prediction:**

\[
\mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \xrightarrow{\text{dynamics model}} \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1})
\]

\[
x_{k|k-1} = F_{k|k-1}x_{k-1|k-1}
\]

\[
P_{k|k-1} = F_{k|k-1}P_{k-1|k-1}F_{k|k-1}^\top + D_{k|k-1}
\]

**filtering:**

\[
\mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \xrightarrow{\text{current measurement } z_k} \mathcal{N}(x_k; x_{k|k}, P_{k|k})
\]

\[
x_{k|k} = x_{k|k-1} + W_{k|k-1} \nu_{k|k-1}, \quad \nu_{k|k-1} = z_k - H_k x_{k|k-1}
\]

\[
P_{k|k} = P_{k|k-1} - W_{k|k-1} S_{k|k-1} W_{k|k-1}^\top, \quad S_{k|k-1} = H_k P_{k|k-1} H_k^\top + R_k
\]

\[
W_{k|k-1} = P_{k|k-1} H_k^\top S_{k|k-1}^{-1}, \quad '\text{KALMAN gain matrix}'
\]
S_k Sensors Producing Target Measurement at the Same Time

One possibility:

\[ H_k x_k = \begin{pmatrix} H_k^1 \\ \vdots \\ H_k^{S_k} \end{pmatrix} x_k, \quad R_k = \text{diag}[R_k^1, \ldots, R_k^{S_k}] \]

Alternatively, provided that \( H_k^i = H_k, \ i = 1, \ldots, S_k \):

\[
p(z_k^1, z_k^2, \ldots, z_k^{S_k} | x_k) = \prod_{s=1}^{S_k} p(z_k^s | x_k) \quad \text{independent sensors}
\]

\[
= \prod_{s=1}^{S_k} \mathcal{N}(z_k^s; H_k x_k, R_k^s) \propto \mathcal{N}(z_k; H_k x_k, R_k)
\]

with:

\[
z_k = R_k \sum_{s=1}^{S_k} (R_k^s)^{-1} z_k^s, \quad R_k = \left( \sum_{s=1}^{S_k} (R_k^s)^{-1} \right)^{-1}
\]
Retrodiction: How to calculate the pdf $p(x_l|Z^k)$?

Consider the past: $l < k$!

an observation:

$$p(x_l|Z^k) = \int d_{x_{l+1}} p(x_l, x_{l+1}|Z^k)$$
Retrodiction: How to calculate the pdf \( p(x_l | \mathcal{Z}^k) \)?

Consider the past: \( l < k \)!

an observation:

\[
p(x_l | \mathcal{Z}^k) = \int d x_{l+1} \ p(x_l, x_{l+1} | \mathcal{Z}^k) = \int d x_{l+1} \ p(x_l | x_{l+1}, \mathcal{Z}^k) \underbrace{p(x_{l+1} | \mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}
\]
Retrodiction: How to calculate the pdf $p(x_l | \mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p(x_l | \mathcal{Z}^k) = \int dx_{l+1} p(x_l, x_{l+1} | \mathcal{Z}^k) = \int dx_{l+1} p(x_l | x_{l+1}, \mathcal{Z}^k) \underbrace{p(x_{l+1} | \mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

$$p(x_l | x_{l+1}, \mathcal{Z}^k) = \frac{p(Z_k, \ldots, Z_{l+1} | x_{l+1}, x_l, \mathcal{Z}^l) p(x_l | x_{l+1}, \mathcal{Z}^l)}{\int dx_l p(Z_k, \ldots, Z_{l+1} | x_{l+1}, x_l, \mathcal{Z}^l) p(x_l | x_{l+1}, \mathcal{Z}^l)} = p(x_l | x_{l+1}, \mathcal{Z}^l)$$
Retrodiction: How to calculate the pdf $p(x_l|\mathcal{Z}^k)$?

Consider the past: $l < k$!

an observation:

$$p(x_l|\mathcal{Z}^k) = \int dx_{l+1} p(x_l, x_{l+1}|\mathcal{Z}^k) = \int dx_{l+1} p(x_l|x_{l+1}, \mathcal{Z}^k) \underbrace{p(x_{l+1}|\mathcal{Z}^k)}_{\text{retrodiction: } t_{l+1}}$$

$$p(x_l|x_{l+1}, \mathcal{Z}^k) = p(x_l|x_{l+1}, \mathcal{Z}^l) = \frac{p(x_{l+1}|x_l) p(x_l|\mathcal{Z}^l)}{\int dx_l \underbrace{p(x_{l+1}|x_l)}_{\text{dynamics model}} \underbrace{p(x_l|\mathcal{Z}^l)}_{\text{filtering } t_l}}$$
Retrodiction: How to calculate the pdf $p(x_l|Z^k)$?

Consider the past: $l < k$!

an observation:

\[
p(x_l|Z^k) = \int dx_{l+1} p(x_l, x_{l+1}|Z^k) = \int dx_{l+1} \frac{p(x_{l+1}|x_l) p(x_l|Z^l)}{\int dx_l p(x_{l+1}|x_l) p(x_l|Z^l)} p(x_l|Z^l)
\]

- $p(x_{l+1}|Z^k)$ retrodiction: last iteration step
- $p(x_k|x_{k-1})$ dynamic object behavior
- $p(x_l|Z^l)$ filtering at the time considered

- GAUSSians, GAUSSian mixtures: Exploit product formula!
- linear GAUSSian likelihood/dynamics: Rauch-Tung-Striebel smoothing
Exercise 6.1  Derive the *Rauch-Tung-Striebel* formulae

by using the Kalman filter assumptions

and the product formula (twice)!

\[
\begin{align*}
\text{retrodiction:} & \quad \mathcal{N}\left( x_l; x_{l|k}, P_{l|k} \right) \xrightarrow{\text{filtering, prediction}} \mathcal{N}\left( x_{l+1}; x_{l+1|k}, P_{l+1|k} \right) \\
x_{l|k} &= x_{l|l} + W_{l|l+1}(x_{l+1|k} - x_{l+1|l}), \\
P_{l|k} &= P_{l|l} + W_{l|l+1}(P_{l+1|k} - P_{l+1|l})W_{l|l+1}^{\top} \\
W_{l|l+1} &= P_{l|l}F_{l+1|l}P_{l+1|l}^{-1}
\end{align*}
\]
Kalman filter: linear Gaussian likelihood/dynamics, \( x_k = (r^T_k, \dot{r}^T_k, \ddot{r}^T_k) \), \( Z^k = \{ z_k, Z^{k-1} \} \)

**initiation:**
\[ p(x_0) = \mathcal{N}(x_0; x_{0|0}, P_{0|0}), \quad \text{initial ignorance:} \quad P_{0|0} \text{ ‘large’} \]

**prediction:**
\[ \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \xrightarrow{\text{dynamics model}} \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \]
\[ x_{k|k-1} = F_{k|k-1} x_{k-1|k-1} \]
\[ P_{k|k-1} = F_{k|k-1} P_{k-1|k-1} F_{k|k-1}^\top + D_{k|k-1} \]

**filtering:**
\[ \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \xrightarrow{\text{current measurement} z_k} \mathcal{N}(x_k; x_{k|k}, P_{k|k}) \]
\[ x_{k|k} = x_{k|k-1} + W_{k|k-1} \nu_{k|k-1}, \quad \nu_{k|k-1} = z_k - H_k x_{k|k-1} \]
\[ P_{k|k} = P_{k|k-1} - W_{k|k-1} S_{k|k-1} W_{k|k-1}^\top, \quad S_{k|k-1} = H_k P_{k-1|k-1} H_k^\top + R_k \]
\[ W_{k|k-1} = P_{k|k-1} H_k^\top S_{k|k-1}^{-1} \]

**retridiction:**
\[ \mathcal{N}(x_l; x_{l|k}, P_{l|k}) \xrightarrow{\text{filtering, prediction}} \mathcal{N}(x_{l+1}; x_{l+1|k}, P_{l+1|k}) \]
\[ x_{l|k} = x_{l|l} + W_{l|l+1} (x_{l+1|k} - x_{l+1|l}), \quad W_{l|l+1} = P_{l|l} F_{l+1|l}^\top P_{l+1|l}^{-1} \]
\[ P_{l|k} = P_{l|l} + W_{l|l+1} (P_{l+1|k} - P_{l+1|l}) W_{l|l+1}^\top \]

**Exercise 6.2** Implement the Rauch-Tung-Striebel formulae in your simulator!
Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(x_l|Z^k)$ and $p(x_{l+1}|Z^k)$ based on the evolution model:

$$p(x_{l+\theta}|Z^k) = \int dx_{l+1} \ p(x_{l+\theta}, x_{l+1}|Z^k)$$

$$= \int dx_{l+1} \ p(x_{l+\theta}|x_{l+1}, Z^k) \ p(x_{l+1}|Z^k)$$
Continuous Time Retrodiction for $t_l < t_{l+\theta} < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(x_l|Z^k)$ and $p(x_{l+1}|Z^k)$ based on the evolution model:

$$p(x_{l+\theta}|Z^k) = \int dx_{l+1} \ p(x_{l+\theta}, x_{l+1}|Z^k)$$
$$= \int dx_{l+1} \ p(x_{l+\theta}|x_{l+1}, Z^k) \ p(x_{l+1}|Z^k)$$

where:

$$p(x_{l+\theta}|x_{l+1}, Z^k) = \frac{p(x_{l+1}|x_{l+\theta}) \ p(x_{l+\theta}|Z^l)}{\int dx_{l+\theta} \ p(x_{l+1}|x_{l+\theta}) \ p(x_{l+\theta}|Z^l)}$$

with:

$$p(x_{l+1}|x_{l+\theta}) = \mathcal{N}(x_{l+1}; \ F_{l+1|l+\theta}x_{l+\theta}, \ D_{l+1|l+\theta})$$
$$p(x_{l+\theta}|Z^l) = \int dx_l \ p(x_{l+\theta}|x_l) \ p(x_l|Z^l)$$
$$p(x_{l+1}|Z^l) = \int dx_{l+\theta} \ p(x_{l+1}|x_{l+\theta}) \ p(x_{l+\theta}|Z^l)$$
$$= \mathcal{N}(x_{l+1}; x_{l+1|l}, \ P_{l+1|l})$$
Continuous Time Retrodiction for $t_l < t_l + \theta < t_{l+1}$ with $0 < \theta < 1$

Interpolate between $p(x_l|Z^k)$ and $p(x_{l+1}|Z^k)$ based on the evolution model:

$$p(x_{l+\theta}|Z^k) = \int dx_{l+1} p(x_{l+\theta}, x_{l+1}|Z^k)$$
$$= \int dx_{l+1} p(x_{l+\theta}|x_{l+1}, Z^k) p(x_{l+1}|Z^k)$$

where:

$$p(x_{l+\theta}|x_{l+1}, Z^k) = \frac{p(x_{l+1}|x_{l+\theta}) p(x_{l+\theta}|Z^l)}{\int dx_{l+\theta} p(x_{l+1}|x_{l+\theta}) p(x_{l+\theta}|Z^l)}$$

with:

$$p(x_{l+1}|x_{l+\theta}) = \mathcal{N}(x_{l+1}; F_{l+1|l+\theta}x_{l+\theta}, D_{l+1|l+\theta})$$

$$p(x_{l+\theta}|Z^l) = \int dx_l p(x_{l+\theta}|x_l) p(x_l|Z^l)$$

$$p(x_{l+1}|Z^l) = \int dx_{l+\theta} p(x_{l+1}|x_{l+\theta}) p(x_{l+\theta}|Z^l)$$
$$= \mathcal{N}(x_{l+1}; x_{l+1|l}, P_{l+1|l})$$

Looks like a Kalman filtering update!
\[ p(x_{l+\theta}|x_{l+1}, \mathcal{Z}^k) \propto p(x_{l+1}|x_{l+\theta}) p(x_{l+\theta}|\mathcal{Z}^l) \quad \text{Looks like filtering!} \]

\[ p(x_{l+\theta}|\mathcal{Z}^k) = \int dx_{l+1} p(x_{l+\theta}|x_{l+1}, \mathcal{Z}^k) p(x_{l+1}|\mathcal{Z}^k) \quad \text{Looks like prediction!} \]
\[ p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \propto p(\mathbf{x}_{l+1}|\mathbf{x}_{l+\theta}) \, p(\mathbf{x}_{l+\theta}|\mathcal{Z}^l) \quad \text{Looks like filtering!} \]
\[ = \mathcal{N}(\mathbf{x}_{l+\theta}; \mathbf{a}_{l+\theta|l+1}, \Delta_{l+\theta|l+1}) \]

\[ \mathbf{a}_{l+\theta|l+1} = \mathbf{x}_{l+\theta|l} + \Phi_{l+\theta|l+1}(\mathbf{x}_{l+1} - \mathbf{F}_{l+\theta|l+1}\mathbf{x}_{l+\theta|l}) \]
\[ = \mathbf{x}_{l+\theta|l} - \Phi_{l+\theta|l+1}\mathbf{x}_{l+1|l} + \Phi_{l+\theta|l+1}\mathbf{x}_{l+1} \]

\[ \Delta_{l+\theta|l+1} = P_{l+\theta|l} - \Phi_{l+\theta|l+1}P_{l+1|l}\Phi_{l+\theta|l+1}^\top \]
\[ \Phi_{l+\theta|l+1} = P_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^\top P_{l+1|l}^{\theta-1} \]
\[ P_{l+1|l} = \mathbf{F}_{l+1|l+\theta}P_{l+\theta|l}\mathbf{F}_{l+1|l+\theta}^\top + D_{l+1|l+\theta} \]

\[ p(\mathbf{x}_{l+\theta}|\mathcal{Z}^k) = \int d\mathbf{x}_{l+1} \, p(\mathbf{x}_{l+\theta}|\mathbf{x}_{l+1}, \mathcal{Z}^k) \, p(\mathbf{x}_{l+1}|\mathcal{Z}^k) \quad \text{Looks like prediction!} \]
\[ p(x_{l+\theta} \mid x_{l+1}, Z^k) \propto p(x_{l+1} \mid x_{l+\theta}) p(x_{l+\theta} \mid Z^l) \quad \text{Looks like filtering!} \]

\[ = \mathcal{N}(x_{l+\theta} \mid a_{l+\theta|l+1}, \Delta_{l+\theta|l+1}) \]

\[ = \mathcal{N}(b_{l+\theta|l+1} \mid \Phi_{l+\theta|l+1}x_{l+1}, \Delta_{l+\theta|l+1}) \]

\[ a_{l+\theta|l+1} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}(x_{l+1} - F_{l+1|l+\theta}x_{l+\theta|l}) \]

\[ = x_{l+\theta|l} - \Phi_{l+\theta|l+1}x_{l+1} + \Phi_{l+\theta|l+1}x_{l+1} \]

\[ b_{l+\theta|l+1} = x_{l+\theta} - x_{l+\theta|l} + \Phi_{l+\theta|l+1}x_{l+1} \]

\[ \Delta_{l+\theta|l+1} = P_{l+\theta|l} - \Phi_{l+\theta|l+1}P_{l+1|l} \Phi_{l+\theta|l+1}^T \]

\[ \Phi_{l+\theta|l+1} = P_{l+\theta|l}F_{l+1|l+\theta}^{-1}P_{l+1|l} \]

\[ P_{l+1|l} = F_{l+1|l+\theta}P_{l+\theta|l}F_{l+1|l+\theta}^T + D_{l+1|l+\theta}. \]

\[ p(x_{l+\theta} \mid Z^k) = \int dx_{l+1} p(x_{l+\theta} \mid x_{l+1}, Z^k) p(x_{l+1} \mid Z^k) \quad \text{Looks like prediction!} \]
\[ p(x_{l+\theta|x_{l+1}}, z^k) \propto p(x_{l+1|x_{l+\theta}}) p(x_{l+\theta|z^l}) \quad \text{Looks like filtering!} \]

\[ = \mathcal{N}(x_{l+\theta}; a_{l+\theta|l+1}, \Delta_{l+\theta|l+1}) \]

\[ = \mathcal{N}(b_{l+\theta|l+1}; \Phi_{l+\theta|l+1}x_{l+1}, \Delta_{l+\theta|l+1}) \]

\[ a_{l+\theta|l+1} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}(x_{l+1} - F_{l+1|l+\theta}x_{l+\theta|l}) \]

\[ = x_{l+\theta|l} - \Phi_{l+\theta|l+1}x_{l+1} + \Phi_{l+\theta|l+1}x_{l+1} \]

\[ b_{l+\theta|l+1} = x_{l+\theta} - x_{l+\theta|l} + \Phi_{l+\theta|l+1}x_{l+1|l} \]

\[ \Delta_{l+\theta|l+1} = P_{l+\theta|l} - \Phi_{l+\theta|l+1}P_{l+1|l} \Phi_{l+\theta|l+1}^T \]

\[ \Phi_{l+\theta|l+1} = P_{l+\theta|l}F_{l+1|l+\theta}P_{l+1|l}^{-1} \]

\[ P_{l+1|l} = F_{l+1|l+\theta}P_{l+\theta|l} \Phi_{l+\theta|l+1}^T + D_{l+1|l+\theta}. \]

\[ p(x_{l+\theta|z^k}) = \int dx_{l+1} p(x_{l+\theta|x_{l+1}}, z^k) p(x_{l+1|z^k}) \quad \text{Looks like prediction!} \]

\[ = \mathcal{N}(x_{l+\theta}; x_{l+\theta|k}, x_{l+\theta|k}) \]

\[ x_{l+\theta|k} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}(x_{l+1|k} - x_{l+1|l}) \]

\[ P_{l+\theta|k} = x_{l+\theta|l} + \Phi_{l+\theta|l+1}(P_{l+1|k} - P_{l+1|l}) \Phi_{l+\theta|l+1}^T \]
Kalman filter: linear GAUSSian likelihood/dynamics, \( x_k = (r_k^T, \dot{r}_k^T, \ddot{r}_k^T)^T, \) \( Z^k = \{ z_k, Z^{k-1} \} \)

**initiation:** 
\[ p(x_0) = \mathcal{N}(x_0; x_{0|0}, P_{0|0}) \]
initial ignorance: \( P_{0|0} \) 'large'

**prediction:** 
\[ \mathcal{N}(x_{k-1}; x_{k-1|k-1}, P_{k-1|k-1}) \xrightarrow{\text{dynamics model}} \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \]
\[ x_{k|k-1} = F_{k|k-1} x_{k-1|k-1} \]
\[ P_{k|k-1} = F_{k|k-1} P_{k-1|k-1} F_{k|k-1}^\top + D_{k|k-1} \]

**filtering:** 
\[ \mathcal{N}(x_k; x_{k|k-1}, P_{k|k-1}) \xrightarrow{\text{current measurement } z_k} \mathcal{N}(x_k; x_{k|k}, P_{k|k}) \]
\[ x_{k|k} = x_{k|k-1} + W_{k|k-1} \nu_{k|k-1}, \]
\[ \nu_{k|k-1} = z_k - H_k x_{k|k-1} \]
\[ P_{k|k} = P_{k|k-1} - W_{k|k-1} S_{k|k-1} W_{k|k-1}^\top, \]
\[ S_{k|k-1} = H_k P_{k|k-1} H_k^\top + R_k \]
\[ W_{k|k-1} = P_{k|k-1} H_k S_{k|k-1}^{-1} \]

‘KALMAN gain matrix’

**retrodiction:** 
\[ \mathcal{N}(x_l; x_{l|k}, P_{l|k}) \xleftarrow{\text{filtering, prediction}} \mathcal{N}(x_{l+1}; x_{l+1|k}, P_{l+1|k}) \]
\[ x_{l|k} = x_{l|l} + W_{l|l+1}(x_{l+1|k} - x_{l+1|l}), \]
\[ W_{l|l+1} = P_{l|l} F_{l+1|l}^\top P_{l+1|l}^{-1} \]
\[ P_{l|k} = P_{l|l} + W_{l|l+1}(P_{l+1|k} - P_{l+1|l}) W_{l|l+1}^\top \]
Sensor data: range, azimuth, range-rate

Coordinates: Sensor data → *polar*, object evolution → *Cartesian*

Dynamics system:

\[
\begin{align*}
    x^d &= (x, y, \dot{x}, \dot{y}) \\
    p(x^d_{k-1} | Z^{k-1}) &\xrightarrow{\text{Dynamics}} p(x^d_k | Z^k) \quad \nu \quad \text{scan } k - 1 \\
    t_{d \leftarrow s} &\uparrow \\
    \text{Sensor system:} \\
    x^s &= (r, \varphi, \dot{r}, \dot{\varphi}) \\
    p(x^s_{k-1} | Z^{k-1}) &\xrightarrow{\text{Sensor}} p(x^s_k | Z^k) \quad \nu \quad \text{scan } k \quad \text{scan } k \\
    t_{s \leftarrow d} &\downarrow \\
\end{align*}
\]

\text{*non-linear* coordinate transformations:}

\[
\begin{align*}
    t_{d \leftarrow s}[x^s] &= \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} r \cos \varphi \\ r \sin \varphi \\ \dot{r} \cos \varphi - \dot{\varphi} \sin \varphi \\ \dot{r} \sin \varphi + r \varphi \cos \varphi \end{pmatrix} \\
    t_{s \leftarrow d}[x^d] &= \begin{pmatrix} r \\ \varphi \\ \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{x^2+y^2}}{\arctan \frac{y}{x}} \\ \frac{(x \dot{x} + y \dot{y})}{\sqrt{x^2+y^2}} \\ \frac{(x \dot{y} - y \dot{x})}{\sqrt{x^2+y^2}} \end{pmatrix}
\end{align*}
\]
Extended *Kalman* filter: be wise - linearize!

non-linear transformations: Taylor expansion up to 1st order

around $\mathbf{x}^s_{k|k}$ (filtering):

$t_{d\leftarrow s}[\mathbf{x}^s_k] \approx t_{d\leftarrow s}[\mathbf{x}^s_{k|k}] + T_{d\leftarrow s}[\mathbf{x}^s_{k|k}](\mathbf{x}^s_k - \mathbf{x}^s_{k|k})$

mit: $T_{d\leftarrow s}[\mathbf{x}^s_{k|k}] = \partial t_{d\leftarrow s}[\mathbf{x}^s_{k|k}] / \partial \mathbf{x}^s_{k|k}$ (Jacobian)

around $\mathbf{x}^d_{k|k-1}$ (Prediction):

$t_{s\leftarrow d}[\mathbf{x}^d_k] \approx t_{s\leftarrow d}[\mathbf{x}^d_{k|k-1}] + T_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}](\mathbf{x}^d_k - \mathbf{x}^d_{k|k-1})$

with: $T_{s\leftarrow d} = \partial t_{d\leftarrow s}[\mathbf{x}^d_{k|k-1}] / \partial \mathbf{x}^d_{k|k-1}$

affine transformation of Gaussian random variables:

$$\mathcal{N}(x; \mathbf{x}, \mathbf{X}) \xrightarrow{y = a + A x} \mathcal{N}(y; a + A x, A \mathbf{X} A^\top)$$
A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

\[ p(z_k | Z^{k-1}) = \int dx_k p(z_k, x_k | Z^{k-1}) = \int dx_k p(z_k | x_k) p(x_k | Z^{k-1}) \]
A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

\[
p(z_k | Z^{k-1}) = \int dx_k \ p(z_k, x_k | Z^{k-1}) = \int dx_k \ p(z_k | x_k) \ p(x_k | Z^{k-1})
\]

\[
= \int dx_k \ N(z_k; H_k x_k, R_k) \ N(x_k; x_k|k-1, P_{k|k-1})
\]

likelihood: sensor model \hspace{1cm} \text{prediction at time } t_k
A side Result: *Expected* Measurements

innovation statistics, expectation gates, gating

\[
p(z_k | z^{k-1}) = \int dx_k \ p(z_k, x_k | z^{k-1}) = \int dx_k \ p(z_k | x_k) \ p(x_k | z^{k-1})
\]

\[
= \int dx_k \ \mathcal{N}(z_k; H_k x_k, R_k) \ \mathcal{N}(x_k; x_k|k-1, P_{k|k-1})
\]

\[
= \mathcal{N}(z_k; H_k x_k|k-1, S_{k|k-1}) \quad \text{(product formula)}
\]

innovation:
\[
\nu_{k|k-1} = z_k - H_k x_k|k-1,
\]

innovation covariance:
\[
S_{k|k-1} = H_k P_{k|k-1} H_k^\top + R_k
\]

expectation gate:
\[
\nu_{k|k-1}^\top S_{k|k-1}^{-1} \nu_{k|k-1} \leq \lambda(P_C)
\]

**Mahalanobis** ellipsoid containing \(z_k\) with certain probability \(P_c\)

Choose \(\lambda(P_c)\) ("gating parameter") properly!

Can be looked up in a \(\chi^2\)-table - discussed later!
Sensor data of uncertain origin

- prediction: \( x_{k|k-1}, P_{k|k-1} \) (dynamics)
- innovation: \( \nu_k = z_k - Hx_{k|k-1} \), white
- Mahalanobis norm: \( ||\nu_k||^2 = \nu_k^T S_k^{-1} \nu_k \)
- expected plot: \( z_k \sim N(Hx_{k|k-1}, S_k) \)
- \( \nu_k \sim N(0, S_k), S_k = HP_{k|k-1}H^T + R \)
- gating: \( ||\nu_k|| < \lambda, P_c(\lambda) \) correlation prob.

missing/false plots, measurement errors, scan rate, agile targets: large gates
A Generic Tracking and Sensor Data Fusion System

Tracking & Fusion System

Sensor System

Sensing Hardware:
- Received Waveforms
Detection Process:
- Data Rate Reduction
Signal Processing:
- Parameter Estimation

A Priori Knowledge:
- Sensor Performance
- Object Characteristics
- Object Environment

Sensor Data to Track Association

Track Initiation:
- Multiple Frame Track Extraction

Track Maintenance:
- Prediction, Filtering Retrodiction

Track File Storage

Track Processing:
- Track Cancellation
- Object Classification / ID
- Track-to-Track Fusion

Man-Machine Interface:
- Object Representation
- Displaying Functions
- Interaction Facilities

Sensor System

Sensor System

Sensor Data Fusion - Methods and Applications, 6th Lecture on May 28, 2018
Description of the Detection Process

**Detector:** receives signals and decides on object existence

**Processor:** processes detected signals and produces measurements

‘\(D\)': detector detects an object

\(D\): object actually existent
Description of the Detection Process

**Detector:** receives signals and decides on object existence

**Processor:** processes detected signals and produces measurements

\('D\)': detector detects an object
\(D\): object actually existent

error of 1. kind: \(P_I = P(\neg 'D' | D)\)

error of 2. kind: \(P_{II} = P('D' | \neg D)\)

measure of detection performance: \(P_D = P('D' | D)\)

detector properties characterized by two parameters:

- detection probability \(P_D = 1 - P_I\)
- false alarm probability \(P_F = P_{II}\)
**Description of the Detection Process**

**Detector**: receives signals and decides on object existence

**Processor**: processes detected signals and produces measurements

‘D’: detector detects an object  

\[ P_I = P(\neg \text{‘D’}|D) \]  

error of 1. kind

\[ P_{II} = P(\text{‘D’}|\neg D) \]  

error of 2. kind

\[ P_D = P(\text{‘D’}|D) \]  

measure of detection performance

\[ P_D = 1 - P_I \]  

detection probability

\[ P_F = P_{II} \]  

false alarm probability

example (Swerling I model):  

\[ P_D = P_D(P_F, \text{SNR}) = P_F^{1/(1+\text{SNR})} \]

**detector design**: Maximize detection probability \( P_D \)  

for a given, predefined false alarm probability \( P_F \)!
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

\[ n_k + 1 \text{ possible interpretations of the sensor data } Z_k = \{z^j_k\}_{j=1}^{n_k}! \]

- \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
- \(E_j, j = 1, \ldots, n_k\): Object detected; \(z^j_k\) is object measurement; \(n_k - 1\) false measurements

Consider the interpretations in the likelihood function \(p(Z_k, n_k | x_k)!\)
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

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\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k) \quad D = \text{“object was detected”}
\]
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

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\]

\[
= p(Z_k, n_k, \neg D|x_k) \frac{P(\neg D|x_k)}{1 - P_D} + p(Z_k, n_k, D|x_k) \frac{P(D|x_k)}{P_D}
\]

sensor parameter: detection probability \(P_D\)
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

\[ n_k + 1 \text{ possible interpretations of the sensor data } Z_k = \{z_k^j\}_{j=1}^{n_k}! \]

- \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
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\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k) \quad D = \text{“object was detected”} \\
= p(Z_k, n_k, \neg D, x_k) P(\neg D|x_k) + p(Z_k, n_k| D, x_k) p(D|x_k) \\
= p(Z_k| n_k, \neg D, x_k) p(n_k|\neg D, x_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j| D, x_k) \\
= p_F(n_k) \quad \text{false measurements: Poisson distributed in #, uniformly distributed in the FoV} \]
Modeling of False Measurements (FM)

- Probability of having $n$ FM: $p_F(n)$

  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    \[
    \bar{n} = \rho_F |\text{FoV}|, \quad \text{false measurement density } \rho_F \text{ (perhaps not constant)}
    \]
Modeling of False Measurements (FM)

- **Probability of having** \( n \) FM: \( p_F(n) \)
  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    \[
    \tilde{n} = \rho_F |\text{FoV}|, \quad \text{false measurement density } \rho_F \text{ (perhaps not constant)}
    \]
  - assumption: \( n \) is a Poisson distributed RV with
    \[
    p_F(n) = \frac{\tilde{n}^n}{n!} e^{-\tilde{n}}
    \]
Modeling of False Measurements (FM)

- Probability of having $n$ FM: $p_F(n)$
  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    $$\bar{n} = \rho_F|\text{FoV}|,$$
    false measurement density $\rho_F$ (perhaps not constant)
  - assumption: $n$ is a Poisson distributed RV with
    $$p_F(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}$$
    expectation: $\mathbb{E}[n] = \bar{n}$, variance: $\mathbb{V}[n] = \bar{n}$
normalization: \( \sum_{n=0}^{\infty} p_F(n) = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = e^{-\bar{n}} \bar{e}^\bar{n} = 1 \)

expectation: \( \mathbb{E}[n] = e^{-\bar{n}} \sum_{n=0}^{\infty} n \frac{\bar{n}^n}{n!} = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} = \bar{n} e^{-\bar{n}} \sum_{n=1}^{\infty} \frac{\bar{n}^{n-1}}{(n-1)!} = \bar{n} \)

variance: \( \text{Var}[n] = \mathbb{E}[(n - \bar{n})^2] = \mathbb{E}[n^2] - \bar{n}^2 = \ldots \text{exercise!} \ldots = \bar{n} \)
Modeling of False Measurements (FM)

- **Probability of having** \( n \) **FM:** \( p_F(n) \)
  
  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    \[ \bar{n} = \rho_F|\text{FoV}|, \text{ false measurement density } \rho_F \text{ (perhaps not constant)} \]
  
  - assumption: \( n \) is a Poisson distributed RV with
    \[ p_F(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}} \]
    
    expectation: \( \mathbb{E}[n] = \bar{n} \), variance: \( \mathbb{V}[n] = \bar{n} \)

- **Distribution of FM in the Field of View:** \( p(z_1^f, \ldots, z_n^f|\text{FoV}) \)
  
  - FM mutually independent:
    \[ p(z_1^f, \ldots, z_n^f|\text{FoV}) = \prod_{i=1}^{n} p(z_i^f|\text{FoV}) \]
Modeling of False Measurements (FM)

- **Probability of having** $n$ FM: $p_F(n)$
  - mean number of FM in the ‘Field of View’ (FoV) of a sensor:
    $$\bar{n} = \rho_F |\text{FoV}|$$, false measurement density $\rho_F$ (perhaps not constant)
  - assumption: $n$ is a Poisson distributed RV with
    $$p_F(n) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}$$
    expectation: $\mathbb{E}[n] = \bar{n}$, variance: $\mathbb{V}[n] = \bar{n}$

- **Distribution of FM in the Field of View:** $p(z^f_1, \ldots, z^f_n | \text{FoV})$
  - FM mutually independent: $p(z^f_1, \ldots, z^f_n | \text{FoV}) = \prod_{i=1}^{n} p(z^f_i | \text{FoV})$
  - uniformly distributed in the FoV: $p(z^f_i | \text{FoV}) = |\text{FoV}|^{-1}$ (often!)
ambiguous sensor data \( (P_D < 1, \rho_F > 0) \)

\[ n_k + 1 \text{ possible interpretations of the sensor data } Z_k = \{z^j_k\}_{j=1}^{n_k}! \]

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\[
p(Z_k, n_k|x_k) = p(Z_k, n_k, \neg D|x_k) + p(Z_k, n_k, D|x_k) \quad D = "\text{object was detected}" \\
= p(Z_k, n_k, \neg D, x_k) p(\neg D|x_k) + p(Z_k, n_k, D, x_k) p(D|x_k) \\
= p(Z_k|n_k, \neg D, x_k) p(n_k|\neg D, x_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, x_k) \\
= |\text{FoV}|^{-n_k} p_F(n_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} \frac{p(Z_k|n_k, j, D, x_k)}{|\text{FoV}|^{-(n_k-1)} N(z^j_k; Hx_k, R)} = 1/n_k \\
= p_F(n_k - 1)
\]

Insert Poisson distribution: \( p_F(n_k) = \frac{(\rho_F|\text{FoV}|)^{-n_k}}{n_k!} e^{-\rho_F|\text{FoV}|} \)
ambiguous sensor data \((P_D < 1, \rho_F > 0)\)

\[ n_k + 1 \text{ possible interpretations of the sensor data } Z_k = \{z_k^j\}_{j=1}^{n_k} ! \]

- \(E_0\): the object was not detected; \(n_k\) false data in the Field of View (FoV)
- \(E_j, j = 1, \ldots, n_k\): Object detected; \(z_k^j\) is object measurement; \(n_k - 1\) false measurements

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\]

\[
= p(Z_k, n_k|\neg D, x_k) P(\neg D|x_k) + p(Z_k, n_k|D, x_k) p(D|x_k)
\]

\[
= p(Z_k|n_k, \neg D, x_k) p(n_k|\neg D, x_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k, n_k, j|D, x_k)
\]

\[
= |\text{FoV}|^{-n_k} p_F(n_k) (1 - P_D) + P_D \sum_{j=1}^{n_k} p(Z_k|n_k, j, D, x_k) p(j|n_k, D) p(n_k|D)
\]

\[
= \frac{e^{-\rho_F|\text{FoV}|}}{n_k !} \rho_F^{n_k-1} \left( (1 - P_D) \rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}(z_k^j; Hx_k, R) \right)
\]

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Likelihood Functions

The likelihood function answers the question:
What does the sensor tell about the state $x$ of the object?

(input: sensor data, sensor model)

- **ideal conditions, one object:** $P_D = 1$, $\rho_F = 0$

  at each time one measurement:

  $p(z_k|x_k) = \mathcal{N}(z_k; Hx_k, R)$

- **real conditions, one object:** $P_D < 1$, $\rho_F > 0$

  at each time $n_k$ measurements $Z_k = \{z_k^1, \ldots, z_k^{n_k}\}$!

  $p(Z_k, n_k|x_k) \propto (1 - P_D)\rho_F + P_D \sum_{j=1}^{n_k} \mathcal{N}(z_k^j; Hx_k, R)$